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A unified theory of zonal flow shears and density corrugations in drift wave turbulence

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Abstract

A unified theory of zonal flow shears and density corrugations in drift wave turbulence is presented. Polarization and density advection beat excitation are studied in combination with modulational response. Noise is driven by two-time flux correlation. While the effective zonal flow eddy viscosity can go negative, the zonal diffusivity is positive definite. There is no inverse cascade of density corrugation. The connection between avalanches and corrugations is discussed. The zonal cross-correlation is identified and calculated. Conditions for alignment of zonal shears and corrugation gradients are determined, and the implications for staircase structure are discussed. We show that the synergy of beat noise and modulational effects is stronger than either alone. Strong zonal flows can be excited well below the modulational instability threshold. In the context of L–H transition, zonal noise quenches turbulence overshoot by eliminating the threshold for zonal flow excitation. The power threshold for L–H transition is lowered.

Keywords: drift wave turbulence, zonal flows, corrugations

(Some figures may appear in colour only in the online journal)

1. Introduction

Appreciation of the role of zonal modes has lead to a paradigm shift in our conception of drift wave turbulence, so much so that we now refer to it as 'drift wave-zonal flow turbulence (DW-ZFT)'. DW-ZFT self-regulates by the interaction of generation—i.e. non-linear transfer to zonal modes with feedback of zonal structures on drift waves by shearing and corrugation [1–3]. DW-ZFT has two components: drift waves ('wavy'-$k_\theta \neq 0$) and zonal modes with $k_\theta = k_z = 0$. Note that symmetry distinguishes zonal from wavy populations. Of course, due to their two directions of symmetry, zonal modes are special, as they are the modes of minimal inertia, transport and damping. It's also important to recall that the zonal mode equations differ in structure from the wavy mode equations, on account of the constraints of symmetry upon electron dynamics. This differs from the corresponding case for geophysical fluids, and renders zonal modes even more important in plasma systems. Symmetry precludes an adiabatic electron response for zonal modes. Thus they are benign repositories for fluctuation energy. Conversion of energy to zonal structures reduces transport and improves confinement. The interaction of zonal and wavy components of DW-ZFT has been encapsulated by an extended predator–prey model [4], which also includes profile structure evolution. Zonal shear flows and profile corrugations have been shown to arrange themselves into long lived quasi-periodic patterns, known as staircases [5–7]. Staircase formation is a striking consequence of inhomogeneous mixing in real space. A theory of DW-ZFT should address both the k-space and real space manifestations of the dynamics.

Zonal modes have been intensively studied by theoretical work [1–3, 8–33], simulations [34–43] and experiment [44–58]. An uncountable infinity of color figures have been generated. Interestingly though, nearly all theoretical models of zonal flow generation divides cleanly into:

(a) calculation of the ZF dielectric, or screening response, with occasional mention of excitation by wavy component beat noise [10, 12]. Indeed, the details and consequences of noise generation have received only cursory examination. This calculation ignores modulational mechanisms;
(b) modulational stability calculations, which consider the response of a pre-existing gas of drift waves to infinitesimal test shears or profile corrugations, but ignore noise emission.

Clearly, this separation is artificial and indeed incorrect. A unified theory of zonal modes is needed. Such a theory must necessarily be formulated at the level of coupled spectrum evolution equations [9, 59], which treat the intrinsically comparable effects of noise emission and coherent response (i.e. turbulent diffusive scattering, where the diffusivity can be negative) on an equal footing. This paper presents the requisite unified theory.

In this paper, we derive coupled spectral equations for zonal flows, density corrugations and the wavy turbulence kinetic energy and internal energy for DW-ZFT in the simple-yet-prototypical Hasegawa–Wakatani model [60, 61]. To eliminate the need to evolve the wavy cross-correlation spectrum $\langle n_k\phi_k^\star\rangle$, we limit this study to the case where $\frac{k_\parallel^2 v_{th}^2}{\omega\nu}\equiv\alpha>1$. In this limit the non-adiabatic electron response is dominated by parallel diffusion, and is laminar. We compute the zonal flow and zonal corrugation incoherent noise, determined by vorticity advection$\left(\vec{\tilde{v}}\cdot\vec{\nabla}\nabla_\perp^2\tilde{\phi}\right)$ and density advection $\left(\vec{\tilde{v}}\cdot\vec{\nabla}\tilde{n}\right)$, respectively. The turbulent flow viscosity and density diffusivity set by the vorticity and density responses, respectively, are also calculated. The analysis shows that the turbulent viscosity can go negative for sufficiently steep energy spectra-$\frac{\partial E}{\partial k_r}<0$ and $\left|\frac{\partial E}{\partial k_r}\right|<\left|\frac{\partial E}{\partial k_r}\right|_{crit}$. This is consistent with the prediction of the wave kinetic theory, and is a condition for nonlocal transfer of energy to large scales. It begs the question of how the system will evolve—i.e. will the turbulence act to relax the strong spectral gradient, thus 'turning off' the familiar negative viscosity phenomenon. The turbulent density diffusivity, however, is seen to be positive definite, suggesting that density perturbations are mixed, and dissipated on small scale. It's important to note here that the turbulent diffusivity is a measure of mixing only for the non-adiabatic density fluctuations. Adiabatic fluctuations$\left(\frac{\delta n_k}{n_0}=\frac{e\delta\phi_k}{T_e}\right)$ are unmixed, since $\hat{z}\times\vec{\nabla}\phi\cdot\vec{\nabla}n=0$ for them.

These developments in the basic physics have important implications for the formation of zonal structures. Polarization beat noise seeds zonal potential at all $k\rho_i<1$ scales. If the scale is modulationally unstable, this seeding results in the growth of strong zonal shears, which then feed back on the primary mode dynamics. This scenario is akin to adding noise to an unstable mode. If the scale is modulationally stable, a noise vs damping competition (akin to that in Brownian motion, which leads to a balance of fluctuation and dissipation) determines the ambient zonal flow levels. For weak damping, these shears can grow quite large, thus rendering the question of modulational instability moot. Note that the presence of noise removes the threshold for zonal mode activity, thus resulting in zonal flow excitation across a broad parameter range.

Corrugations, which are damped by turbulent particle diffusion, also are determined by a noise vs diffusion balance. Thus we see that confronting the DW-ZFT problem now requires one to:

(a) to understand the interplay of noise seeding and negative viscosity, both of which are due to advection of polarization charge;
(b) calculate both shearing-feedback dominated, and fluctuation-dissipation type stationary states. Shearing feedback can regulate the effect of noise excitation on unstable modes;
(c) treat corrugations and flow shears on the same footing. Corrugations can induce random refraction of the wavy modes. The interplay of refraction, shear in $E\times B$ flow and density gradient is determined by the zonal density—vorticity (shear) cross correlation, which is calculated. The importance of the zonal density—vorticity cross correlation has not been appreciated previously;
(d) determine the self-consistent drift-wave spectrum (i.e. not only intensity), and how it compares to the critical slope spectrum of zonal mode dynamics.

In this paper, we present a complete, self-consistent study of zonal mode dynamics. The zonal mode problem is well studied, so it is incumbent upon us to state what is new in this paper. Hence, the novel elements are:

(a) an analysis encompassing both noise generation, modulational instability and their interaction, based upon a systematic spectral closure for zonal shears and corrugations;
(b) the discovery of the forward transfer of internal energy of density corrugations $\sim\left\langle|\overline{n}/n_0|^2\right\rangle$ which occurs along with the familiar inverse transfer of kinetic energy. The zonal flows may exhibit negative viscosity phenomena, but corrugations do not (i.e. diffusivity is positive!);
(c) the realization of the important implications of the zonal cross-correlation $\langle\overline{n}\nabla_\perp^2\overline{\phi}\rangle$ which appear in spectral transfer rates and which governs the phasing of density corrugations and zonal shears. These can be correlated or anti-correlated, depending upon the sign of $\langle\overline{n}\nabla_\perp^2\overline{\phi}\rangle$;
(d) the re-evaluation of predator–prey and L→H model dynamics in light of the role of zonal noise.

Most generally, corrugations can be viewed as real space manifestation of the phase space zonal distribution function and can be calculated from the velocity moments of the the zonal phase space distribution function. Attempts along this line has been made in the [62]. and a transport equation for the zonal phase space distribution function has been derived. However, such a gyro-kinetic description of corrugation has produced no conclusive answer on modulational stability and the dynamics of density and temperature corrugations.

New challenges abound in this classic problem!

The theoretical developments discussed have important practical implications for DW-ZFT evolution, including extended predator–prey models of the $L \to H$ transition. Nonlinear noise excites zonal flows below the modulational instability threshold, thus explaining the broad domain of zonal mode activity. The hard growth/power threshold for zonal flow onset, characteristic of zero noise models, disappears. In $L \to H$ models, turbulence overshoot is consequently eliminated. The steepening of ∇P occurs at lower power, since noise boosts the drive of zonal flows and so necessarily reduces transport. Turbulence and zonal flow energies balance prior to the steepening of ∇P. An improved model of the $L \to H$ transition is discussed at length in this paper.

The remainder of this paper is arranged as follows. Section 2 presents the drift wave—zonal flow system. Section 3 discusses spectral evolution. Section 3.1 calculates the induced diffusion of spectral kinetic energy and internal energy. Section 3.2 presents spectral evolution of zonal intensity. Section 3.3 discusses spectral evolution of density corrugations. Section 3.4 calculates zonal cross-correlations—a quantity heretofore not discussed. Section 4 contrasts the familiar wave kinetic analyses with the spectrum evolution results. The predator–prey model is extended to include the nonlinear noise in section 5. The effect of noise on the L–H transition is addressed in section 6. Section 7 gives conclusions and discussions.

2. Drift wave—zonal mode system

Here we present and discuss the basic model. We consider a plane slab geometry with homogeneous, straight magnetic field in the z direction $\left(\vec{B} = B\hat{z}\right)$ and inhomogeneous density $n_0(x)$. The ions are assumed cold and temperature gradient effects are ignored. The non-linear evolution of dissipative drift wave turbulence is then described by the following 2-field model, due to Hasegawa and Wakatani [60, 61].

$$\frac{d}{dt}\nabla_\perp^2\tilde{\phi} + \tilde{v}_E\cdot\vec{\nabla}\overline{\nabla_\perp^2\phi} = -\chi_e\nabla_\parallel^2\left(\tilde{\phi}-\tilde{n}\right) - \left\{\tilde{\phi},\nabla_\perp^2\tilde{\phi}\right\} + \mu\nabla_\perp^2\nabla_\perp^2\tilde{\phi} \tag{1}$$

$$\frac{d\tilde{n}}{dt} + \tilde{v}_E\cdot\frac{\vec{\nabla}n_0}{n_0} = -\chi_e\nabla_\parallel^2\left(\tilde{\phi}-\tilde{n}\right) - \left\{\tilde{\phi},\tilde{n}\right\} + D_n\nabla_\perp^2\tilde{n}. \tag{2}$$

The above equations (1) and (2) have been written in dimensionless form. Potential and density are normalized as $\tilde{n} = \delta n/n_0$, $\tilde{\phi} = e\delta\phi/T_e$ respectively. Time and space are normalized as $t = \omega_{ci}t$, $x_\perp = x_\perp/\rho_s$. The normalized $E\times B$ velocity is $\tilde{v}_E = \frac{\delta v_E}{c_s} = \hat{z}\times\vec{\nabla}\tilde{\phi}$, $\chi_e = v_{te}^2/\nu_{ei}\Omega_i$ is electron parallel diffusivity, $v_{te} = \sqrt{2T_e/m_e}$ is electron thermal speed, μ is normalized ion viscosity $\mu = \mu_0/\rho_s^2\Omega_i$ and D is normalized collisional particle diffusivity $D_n = D_0/\rho_s^2\Omega_i$. These equations describe non-linear evolutions of vorticity fluctuation $\nabla_\perp^2\tilde{\phi}$ and density fluctuation $\tilde{n}$ which are coupled through parallel electron diffusivity χ_e. The parallel wave length is assumed to be $k_\parallel \sim 1/\text{qR}$ and perpendicular wavelength is $k_\perp\rho_s \sim 1$ so that $k_\parallel \ll k_\perp$ and the equations (1) and (2) describe a quasi-two-dimensional system. In the low collisionality limit, where $\omega_k \ll \chi_e k_\parallel^2$, the electrons are adiabatic i.e. $\tilde{n} = \tilde{\phi}$. Then equations (1) and (2) reduce to the Hasegawa–Mima (H–M), or the Rossby wave equation. In the strongly collisional limit, $\omega_k \gg \chi_e k_\parallel^2$, $\tilde{n}$ and $\tilde{\phi}$ are weakly coupled and evolve separately. Equation (1), for vorticity, reduces to the 2D Navier–Stokes Equation where the vorticity is an active scalar and advected by the $E\times B$ velocity. Equation (2), for the density fluctuation, reduces to a passive scalar equation where the density fluctuation is advected by the same $E\times B$ velocity. Defining the adiabaticity parameter $\alpha \equiv \chi_e k_\parallel^2/\omega_k$, the adiabatic regime corresponds to $\alpha \gg 1$ and the hydrodynamic regime corresponds to $\alpha \ll 1$. The equations for zonal vorticity and zonal density are obtained by zonal averaging (over the directions of symmetry) of the respective fluctuation equations. The zonal density evolution is governed by

$$\frac{d}{dt}\overline{n} = -\frac{\partial}{\partial x}\overline{\tilde{v}_{Ex}\tilde{n}} + D_n\nabla_x^2\overline{n} \tag{3}$$

and the zonal vorticity dynamics is governed by

$$\frac{d}{dt}\nabla_x^2\overline{\phi} = -\frac{\partial}{\partial x}\overline{\tilde{v}_{Ex}\nabla_\perp^2\tilde{\phi}} + \mu\nabla_x^4\overline{\phi} \tag{4}$$

where the first term on the right hand side is the divergence of the vorticity flux. The vorticity flux can be expressed as divergence of Reynolds stress, using the Taylor identity i.e. $\overline{\tilde{v}_{Ex}\nabla_\perp^2\tilde{\phi}} = \frac{\partial}{\partial x}\overline{\tilde{v}_{Ex}\tilde{v}_{Ey}}$. The potential fluctuations are the drift waves governed by the Hasegawa–Wakatani, model for simplicity. The set of equations (1)–(4) constitute a self-consistent model for the coupled drift wave—zonal mode system.

To determine the linear responses, linearize the above equation and taking Fourier transform in the symmetry directions y and z. This yields the dissipative drift wave dispersion relation

$$k_\perp^2\omega_k^2 + i\omega_k\hat{\alpha}\left(1+k_\perp^2\right) - i\omega_{\star e}\hat{\alpha} = 0 \tag{5}$$

where $\hat{\alpha} = \chi_e k_\parallel^2$ and $\omega_{\star e} = (\rho_s/L_n)k_y$ is the drift frequency normalized by Ω_{ci}. The roots of the above dispersion relation equation (5) are:

$$\omega_k = -\frac{i}{2}\hat{\alpha}\left(1+\frac{1}{k_\perp^2}\right) \pm \frac{i}{2}\left\{\hat{\alpha}^2\left(1+\frac{1}{k_\perp^2}\right)^2 - 4i\omega_{\star e}\frac{\hat{\alpha}}{k_\perp^2}\right\}^{1/2}. \tag{6}$$

In the two limiting cases mentioned above, simple expressions for the real frequencies(ω_k^r) and growth rates(γ_k) of the unstable modes are obtained. These are:

(a) *the adiabatic regime* $(\alpha > 1)$: where the real frequency and the growth rate are given by

$$\omega_k^r = \frac{\omega_{\star e}}{1+k_\perp^2} \tag{7}$$

and

$$\gamma_k = \frac{k_\perp^2}{\hat{\alpha}} \frac{\omega_k^{r2}}{1+k_\perp^2} \tag{8}$$

respectively. These modes are conventional drift waves;

(b) *the hydrodynamic regime* $(\alpha < 1)$: In this regime, the growth rate and real frequency are equal, and are given by

$$\omega_k^r = \mathrm{sign}(k_y)\left(\frac{\hat{\alpha}\,|\omega_{\star e}|}{2k_\perp^2}\right)^{1/2} \tag{9}$$

$$\gamma_k = \left(\frac{\hat{\alpha}\,|\omega_{\star e}|}{2k_\perp^2}\right)^{1/2}. \tag{10}$$

3. Spectral evolution

Here we derive spectral equations for the zonal modes and the wave fluctuation energy. Our motivation is to develop the most general inclusive analysis of DW-ZFT. The H-W system in spectral form reads

$$\left(\frac{\partial}{\partial t} + \mu k_\perp^2 + \frac{\hat{\alpha}_k}{k_\perp^2}\right) k_\perp^2 \phi_k - \hat{\alpha}_k n_k = \frac{1}{2}\sum_{\vec{p}+\vec{q}=\vec{k}} \hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)\phi_p\phi_q \tag{11}$$

$$\left(\frac{\partial}{\partial t} + \hat{\alpha}_k\right) n_k + \left(-\hat{\alpha}_k + i\omega_{\star e}\right)\phi_k = \frac{1}{2}\sum_{\vec{p}+\vec{q}=\vec{k}} \hat{z}\cdot\vec{p}\times\vec{q}\left(\phi_p n_q - \phi_q n_p\right). \tag{12}$$

It is straightforward to see that the mode coupling coefficients $M_{kpq}^1 = \hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)/k_\perp^2$ and $M_{kpq}^2 = \hat{z}\cdot\vec{p}\times\vec{q}$ satisfy the the detailed conservation conditions:

$$\sigma_{1k}^{Q_1} M_{kpq}^1 + \sigma_{1p}^{Q_1} M_{pqk}^1 + \sigma_{1q}^{Q_1} M_{qkp}^1 = 0; \qquad Q_1 = (E, Z) \tag{13}$$

and

$$\sigma_{2k}^{Q_2} M_{kpq}^2 + \sigma_{2p}^{Q_2} M_{pqk}^2 + \sigma_{2q}^{Q_2} M_{qkp}^2 = 3\hat{z}\cdot\vec{p}\times\vec{q} \tag{14}$$

where $\sigma_{1k}^{Q_1} = \frac{1}{2}\left(k_\perp^2, k_\perp^4\right)$ and $\sigma_{2k}^{Q_2} = 1$. These symmetry properties guarantee that the polarization non-linear term in equation (11) conserves kinetic energy $E = \sum_k E_k = \sum_k \frac{1}{2}k^2|\phi_k|^2$ and fluid enstrophy $Z = \sum_k Z_k = \sum_k \frac{1}{2}k^4|\phi_k|^2$, and that the convective non-linear term in equation (12) conserves internal energy $E_n = \sum_k E_{nk} = \sum_k \frac{1}{2}|n_k|^2$. In purely adiabatic limit $\hat{\alpha} = \infty$, the density fluctuation is $\tilde{n} = \tilde{\phi}$ and the Hasegawa–Wakatani equations reduce to the H–M equation [63].

$$\left(\frac{\partial}{\partial t} + i\omega_k\right)\phi_k = \frac{1}{2}\sum_{\vec{p}+\vec{q}=\vec{k}} \frac{\hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)}{1+k_\perp^2}\phi_p\phi_q. \tag{15}$$

The mode coupling coefficient $M_{kpq} = \hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)/\left(1+k_\perp^2\right)$ satisfy the detailed conservation condition:

$$\sigma_k^Q M_{kpq} + \sigma_p^Q M_{pqk} + \sigma_q^Q M_{qkp} = 0; \qquad Q = (E, Z) \tag{16}$$

where $\sigma_k^Q = \frac{1}{2}\left(1+k_\perp^2, \left(1+k_\perp^2\right)^2\right)$. This property guarantees non-linear invariance of total energy $E = \sum_k E_k = \sum_k \left(1+k^2\right)|\phi_k|^2$ and total enstrophy $Z = \sum_k Z_k = \sum_k \left(1+k^2\right)^2|\phi_k|^2$ for the H–M system.

For the two-field drift wave turbulence model, the relevant spectra are the kinetic energy spectrum $\left\langle|v_k|^2\right\rangle = k^2\left\langle|\phi|_k^2\right\rangle$, the internal energy spectrum $\left\langle|n_k|^2\right\rangle$ and the cross-correlation spectrum $\langle n_k\phi_k^\star\rangle$. Note that the cross-spectrum is intimately related to the flux i.e. $\Gamma_n = \sum_k -ik_y\langle n_k\phi_k^\star\rangle$ etc. Thus the wave cross-correlation $\langle n_k\phi_k^\star\rangle$ may be thought of as a measure of alignment in drift wave turbulence, much like cross-helicity $\left\langle\vec{v}\cdot\vec{B}\right\rangle$ measures alignment in magnetohydrodynamics (MHD) turbulence. A state of high normalized cross-correlation is one where transport is small, and density mixing or scattering is weak. Likewise, a state of low cross-correlation could support stronger transport and mixing. Zonal fluctuation cross-correlation, discussed later is of broader and different significance. The evolution equation for the kinetic energy spectra is obtained by multiplying the equation (11) by $\phi_k^\star$ and adding the resulting equation with the conjugate of equation (11) multiplied by ϕ_k. Taking a statistical average (denoted by the angular bracket $\langle\rangle$) of the resulting equation yields

$$\left(\frac{\partial}{\partial t} + 2\mu k_\perp^2 + 2\frac{\hat{\alpha}_k}{k_\perp^2}\right) k_\perp^2\left\langle|\phi_k|^2\right\rangle - 2\hat{\alpha}_k\Re\langle n_k\phi_k^\star\rangle = \Re\sum_{\vec{p}+\vec{q}=\vec{k}} \hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)\langle\phi_k^\star\phi_p\phi_q\rangle. \tag{17}$$

Similarly, the internal energy spectrum is obtained as

$$\left(\frac{\partial}{\partial t} + 2\hat{\alpha}_k\right)\left\langle|n_k|^2\right\rangle - 2\hat{\alpha}_k\Re\langle n_k\phi_k^\star\rangle + 2\omega_{\star e}\Im\langle n_k\phi_k^\star\rangle = \Re\sum_{\vec{p}+\vec{q}=\vec{k}} \hat{z}\cdot\vec{p}\times\vec{q}\left(\langle n_k^\star\phi_p n_q\rangle - \langle n_k^\star\phi_q n_p\rangle\right). \tag{18}$$

The evolution equation for the cross correlation spectra is obtained by multiplying the conjugate of equation (11) by n_k and adding the resulting equation to the equation (12) multiplied by $\phi_k^\star$, yielding:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mu k_\perp^2 + \frac{\hat{\alpha}_k}{k_\perp^2} + \hat{\alpha}_k\right) k_\perp^2\langle n_k\phi_k^\star\rangle \\ - \hat{\alpha}_k\left[\left\langle|n_k|^2\right\rangle + k_\perp^2\left\langle|\phi_k|^2\right\rangle\right] + i\omega_{\star e}k_\perp^2\left\langle|\phi_k|^2\right\rangle \\ = \sum_{\vec{p}+\vec{q}=\vec{k}} \hat{z}\cdot\vec{p}\times\vec{q}\left[\left(q^2-p^2\right)\langle n_k\phi_p^\star\phi_q^\star\rangle\right. \\ \left. + k_\perp^2\left(\langle\phi_k^\star\phi_p n_q\rangle - \langle\phi_k^\star\phi_q n_p\rangle\right)\right].\end{aligned} \tag{19}$$

The triplet correlations are determined by the phase coherency of the three modes $\vec{k}$, $\vec{p}$, $\vec{q}$. To first order, in a state of turbulence, this phase coherency is determined by the direct interaction among these three modes in the presence of the stochastic background of all other interactions. Denoting the perturbation in ϕ_k due to this direct interaction by $\delta\phi_k$, the triad correlations are approximated as

$$\langle \phi_k^\star \phi_p \phi_q \rangle = \langle \delta\phi_k^\star \phi_p \phi_q \rangle + \langle \phi_k^\star \delta\phi_p \phi_q \rangle + \langle \phi_k^\star \phi_p \delta\phi_q \rangle \,. \tag{20}$$

The first term on the right hand side in the above equation ultimately leads to non-linear noise or incoherent emission as $\delta\phi_k \sim \phi_p\phi_q$ and so that $\langle \delta\phi_k^\star \phi_p \phi_q \rangle \sim \left\langle |\phi_p|^2 \right\rangle \left\langle |\phi_q|^2 \right\rangle$. Similarly the remaining two terms on the right hand side ultimately represents non-linear relaxation or coherent damping as $\delta\phi_p \sim \phi_k \phi_q^\star$ and so that $\langle \phi_k^\star \delta\phi_p \phi_q \rangle \sim \left\langle |\phi_k|^2 \right\rangle \left\langle |\phi_q|^2 \right\rangle$. The perturbations $\delta\phi_k$ and δn_k are driven by the direct interaction between modes $\vec{p}$ and $\vec{q}$:

$$\left(\frac{\partial}{\partial t} + \eta_k\right) k_\perp^2 \,\delta\phi_k + \hat{\alpha}_k \left(\delta\phi_k - \delta n_k\right) = S_{1k} \tag{21}$$

$$\left(\frac{\partial}{\partial t} + \eta_k\right) \delta n_k + \hat{\alpha}_k \delta n_k + \left(-\hat{\alpha}_k + i\omega_{\star e}\right) \delta\phi_k = S_{2k} \tag{22}$$

where the source terms are given by

$$S_{1k} = \hat{z}\cdot\vec{p}\times\vec{q}\left(q^2 - p^2\right)\phi_p\phi_q \tag{23}$$

$$S_{2k} = \hat{z}\cdot\vec{p}\times\vec{q}\left(\phi_p n_q - \phi_q n_p\right). \tag{24}$$

The solutions of the beat mode equations (21) and (22) are obtained as

$$\delta\phi_k = \int_{-\infty}^{t} dt' e^{-(i\omega_k+\eta_k)(t-t')} \left[a_k S_{1k}(t') + b_k S_{2k}(t')\right] \tag{25}$$

$$\delta n_k = \int_{-\infty}^{t} dt' e^{-(i\omega_k+\eta_k)(t-t')} \left[c_k S_{1k}(t') + d_k S_{2k}(t')\right] \tag{26}$$

where only the dominant virtual mode eigenvalues are retained for simplicity. Heavily damped modes make a small contribution to mediating spectral transfer. The coupling coefficients are given by

$$a_k = \left(1 - \frac{i\omega_k}{\hat{\alpha}_k}\right) b_k; \qquad b_k = \frac{1}{\det(A_k)} \tag{27}$$

$$c_k = \left(1 - \frac{i\omega_k k^2}{\hat{\alpha}_k}\right) a_k; \qquad d_k = \left(1 - \frac{i\omega_k k^2}{\hat{\alpha}_k}\right) b_k \tag{28}$$

where ω_k is the frequency of the linear eigenmode and $det(A_k)$ is given by

$$\det(A_k) = \sqrt{\left(1+k^2\right)^2 - 4i\omega_{\star e}\left(\frac{k^2}{\hat{\alpha}_k}\right)}. \tag{29}$$

In the following the triad correlations are obtained. Using the expression for $\delta\phi_k$ from equation (25) the incoherent part of the potential triad correlation becomes

$$\langle \delta\phi_k^\star \phi_p \phi_q \rangle = \int_{-\infty}^{t} dt' e^{-(-i\omega_k+\eta_k)(t-t')} \left[a_k^\star \langle S_{1k}^\star(t')\phi_p(t)\phi_q(t)\rangle \right.$$
$$\left. + b_k^\star \langle S_{2k}^\star(t')\phi_p(t)\phi_q(t)\rangle\right]$$

where

$$\begin{aligned}
&\langle S_{1k}^\star(t')\phi_p(t)\phi_q(t)\rangle \\
&\quad = \hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)\left\langle \phi_p^\star(t')\phi_q^\star(t')\phi_p(t)\phi_q(t)\right\rangle \\
&\quad = \hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)\left\langle \phi_p^\star(t')\phi_p(t)\right\rangle\left\langle \phi_q^\star(t')\phi_q(t)\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\langle S_{2k}^\star(t')\phi_p(t)\phi_q(t)\rangle \\
&\quad = \hat{z}\cdot\vec{p}\times\vec{q}\left[\left\langle \phi_p^\star(t')n_q^\star(t')\phi_p(t)\phi_q(t)\right\rangle\right. \\
&\qquad \left. - \left\langle \phi_q^\star(t')n_p^\star(t')\phi_p(t)\phi_q(t)\right\rangle\right] \\
&\quad = \hat{z}\cdot\vec{p}\times\vec{q}\left[\left\langle \phi_p^\star(t')\phi_p(t)\right\rangle\left\langle n_q^\star(t')\phi_q(t)\right\rangle\right. \\
&\qquad \left. - \left\langle \phi_q^\star(t')\phi_q(t)\right\rangle\left\langle n_p^\star(t')\phi_p(t)\right\rangle\right].
\end{aligned}$$

Note that the 4th order moment-correlation has been written in terms of products of two second order moment using the assumption of quasi-normal (Gaussian) fluctuation statistics. Similarly using the expression for the potential perturbation $\delta\phi_p$ the coherent part of the triad correlation becomes

$$\langle \phi_k^\star \delta\phi_p \phi_q \rangle = \int_{-\infty}^{t} dt' e^{-(i\omega_p+\eta_p)(t-t')} \left[a_p \langle \phi_k^\star(t) S_{1p}(t')\phi_q(t)\rangle \right.$$
$$\left. + b_p \langle \phi_k^\star(t) S_{2p}(t')\phi_q(t)\rangle\right]$$

where

$$\begin{aligned}
&\langle \phi_k^\star(t) S_{1p}(t')\phi_q(t)\rangle \\
&\quad = \hat{z}\cdot\vec{k}\times\vec{q}\left(k^2-q^2\right)\left\langle \phi_k^\star(t)\phi_q^\star(t')\phi_k(t')\phi_q(t)\right\rangle \\
&\quad = \hat{z}\cdot\vec{p}\times\vec{q}\left(k^2-q^2\right)\left\langle \phi_q^\star(t')\phi_q(t)\right\rangle\left\langle \phi_k^\star(t)\phi_k(t')\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\langle \phi_k^\star(t) S_{2p}(t')\phi_q(t)\rangle \\
&\quad = -\hat{z}\cdot\vec{k}\times\vec{q}\left[\left\langle \phi_k(t')n_q^\star(t')\phi_k^\star(t)\phi_q(t)\right\rangle\right. \\
&\qquad \left. - \left\langle \phi_q^\star(t')n_k(t')\phi_k^\star(t)\phi_q(t)\right\rangle\right] \\
&\quad = -\hat{z}\cdot\vec{p}\times\vec{q}\left[\langle \phi_k(t')\phi_k^\star(t)\rangle\left\langle n_q^\star(t')\phi_q(t)\right\rangle\right. \\
&\qquad \left. - \left\langle \phi_q^\star(t')\phi_q(t)\right\rangle\langle n_k(t')\phi_k^\star(t)\rangle\right].
\end{aligned}$$

Expressing two-time correlations as follows

$$\langle \phi_k^\star(t')\phi_k(t)\rangle = \langle \phi_k^\star(t)\phi_k(t)\rangle\, e^{-(i\omega_k+\eta_k)(t-t')} \tag{30}$$

$$\langle n_k^\star(t')n_k(t)\rangle = \langle n_k^\star(t)n_k(t)\rangle\, e^{-(i\omega_k+\eta_k)(t-t')} \tag{31}$$

$$\langle \phi_k^\star(t')n_k(t)\rangle = \langle \phi_k^\star(t)n_k(t)\rangle\, e^{-(i\omega_k+\eta_k)(t-t')} \tag{32}$$

yields the incoherent emission part as

$$\begin{aligned}\langle \delta\phi_k^\star \phi_p \phi_q\rangle &= \Theta_{kpq}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)\\ &\times\left[\left(q^2-p^2\right)a_k^\star\left\langle|\phi_p|^2\right\rangle\left\langle|\phi_q|^2\right\rangle\right.\\ &\left.+b_k^\star\left(\left\langle|\phi_p|^2\right\rangle\langle n_q^\star\phi_q\rangle-\left\langle|\phi_q|^2\right\rangle\langle n_p^\star\phi_p\rangle\right)\right]\end{aligned} \tag{33}$$

where the triad interaction time is

$$\Theta_{kpq}=\frac{1}{i\left(\omega_p+\omega_q-\omega_k\right)+\eta_k+\eta_p+\eta_q}.$$

Here we note:

$$\begin{aligned}\Theta_{kpq}^{(r)}&=\frac{|\eta_k+\eta_p+\eta_q|}{\left(\omega_p+\omega_q-\omega_k\right)^2+|\eta_k+\eta_p+\eta_q|^2}\\ &=\begin{cases}\pi\delta\left(\omega_p+\omega_q-\omega_k\right) & \text{for}\, \eta< freq.\, mismatch\\ \frac{1}{|\eta_k+\eta_p+\eta_q|} & \text{for}\, \eta> freq.\, mismatch.\end{cases}\end{aligned}$$

Note causality requires the absolute value Θ transitions from its 'weak' to 'strong' turbulence for mismatch frequency $\omega_{MM}\sim\omega_k\sim\eta_k$, which defines the effective Rhines scales [41] $k_\perp^2\rho_s^2\omega_k\sim\eta_k$. Note that the Kubo number in this analysis is $K_u=\tilde{v}\tau_{ac}/\Delta\leqslant 1$. Here $\tilde{v}$ is the fluctuation velocity, τ_{ac} is the autocorrelation time, and Δ is the fluctuation scale. This analysis is a closure theory, which encompasses both 'weak' and 'strong' turbulence limits. For weak turbulence $K_u\sim\tilde{v}\tau_{ac}/\Delta<1$, as $\tilde{v}<v_\star$ (i.e. weak). For strong turbulence $\tau_{ac}\sim\eta_k^{-1}$ and $\eta_k\sim\left(k^2\tilde{v}^2\right)^{1/2}$ so $K_u\sim 1/k\Delta\sim 1$. The coherent part of the triplet correlation becomes

$$\begin{aligned}\langle \phi_k^\star \delta\phi_p \phi_q\rangle &= \Theta_{kpq}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)\\ &\times\left[a_p\left(k^2-q^2\right)\left\langle|\phi_q|^2\right\rangle\left\langle|\phi_k|^2\right\rangle\right.\\ &\left.+b_p\left(\left\langle|\phi_q|^2\right\rangle\langle n_k\phi_k^\star\rangle-\left\langle|\phi_k|^2\right\rangle\langle n_q^\star\phi_q\rangle\right)\right].\end{aligned} \tag{34}$$

Finally the spectral intensity equation becomes:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\mu k_\perp^2+2\frac{\hat{\alpha}_k}{k_\perp^2}\right)k_\perp^2\left\langle|\phi_k|^2\right\rangle-2\hat{\alpha}_k\Re\langle n_k\phi_k^\star\rangle\\ &=\Re\sum_{\vec{p}+\vec{q}=\vec{k}}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\left(q^2-p^2\right)\Theta_{kpq}\\ &\times\left[2a_p\left(k^2-q^2\right)\left\langle|\phi_q|^2\right\rangle\left\langle|\phi_k|^2\right\rangle\right.\\ &\left.+2b_p\left(\left\langle|\phi_q|^2\right\rangle\langle n_k\phi_k^\star\rangle-\left\langle|\phi_k|^2\right\rangle\langle n_q^\star\phi_q\rangle\right)\right]\\ &+\Re\sum_{\vec{p}+\vec{q}=\vec{k}}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\left(q^2-p^2\right)\Theta_{kpq}\left[a_{-k}\left(q^2-p^2\right)\right.\\ &\times\left\langle|\phi_p|^2\right\rangle\left\langle|\phi_q|^2\right\rangle+b_{-k}\left(\left\langle|\phi_p|^2\right\rangle\langle n_q^\star\phi_q\rangle\right.\\ &\left.\left.-\left\langle|\phi_q|^2\right\rangle\langle n_p^\star\phi_p\rangle\right)\right].\end{aligned} \tag{35}$$

The above equation can be written as

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+2\mu k_\perp^2\right)\left\langle|\phi_k|^2\right\rangle+\frac{2\hat{\alpha}_k}{k_\perp^2}\left(1-\frac{\Re\langle n_k\phi_k^\star\rangle}{\left\langle|\phi_k|^2\right\rangle}\right)\left\langle|\phi_k|^2\right\rangle\\ &+2\eta_k\left\langle|\phi_k|^2\right\rangle=F_k\end{aligned} \tag{36}$$

where the eddy damping rate is

$$\begin{aligned}\eta_k&=-\Re\sum_{\vec{p}+\vec{q}=\vec{k}}\frac{1}{k_\perp^2}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\left(q^2-p^2\right)\Theta_{kpq}\\ &\times\left[a_p\left(k^2-q^2\right)+b_p\left(\frac{\langle n_k\phi_k^\star\rangle}{\left\langle|\phi_k|^2\right\rangle}-\frac{\langle n_q^\star\phi_q\rangle}{\left\langle|\phi_q|^2\right\rangle}\right)\right]\left\langle|\phi_q|^2\right\rangle\end{aligned} \tag{37}$$

and the non-linear noise term is

$$\begin{aligned}F_k&=\Re\sum_{\vec{p}+\vec{q}=\vec{k}}\frac{1}{k_\perp^2}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\left(q^2-p^2\right)\Theta_{kpq}\\ &\times\left[a_{-k}\left(q^2-p^2\right)+b_{-k}\left(\frac{\langle n_q^\star\phi_q\rangle}{\left\langle|\phi_q|^2\right\rangle}-\frac{\langle n_p^\star\phi_p\rangle}{\left\langle|\phi_p|^2\right\rangle}\right)\right]\\ &\times\left\langle|\phi_p|^2\right\rangle\left\langle|\phi_q|^2\right\rangle.\end{aligned} \tag{38}$$

From the above equations, one can see that the non-linear transfer of turbulent kinetic energy is dominated by the non-local interactions (i.e. $\vec{p}\neq\vec{q}$), due to presence of the factor $\left(q^2-p^2\right)$. It is also straight forward to see that the cross correlation terms exactly cancel i.e.

$$\frac{\langle n_k\phi_k^\star\rangle}{\left\langle|\phi_k|^2\right\rangle}-\frac{\langle n_q^\star\phi_q\rangle}{\left\langle|\phi_q|^2\right\rangle}=0$$

and

$$\frac{\langle n_q^\star\phi_q\rangle}{\left\langle|\phi_q|^2\right\rangle}-\frac{\langle n_p^\star\phi_p\rangle}{\left\langle|\phi_p|^2\right\rangle}=0$$

in the strongly adiabatic regime $\alpha\rightarrow\infty$, so the above spectral equation reduces to that for the H–M equation.

Similarly, it straightforward to arrive at the following evolution equation for density fluctuation spectrum

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+2D_nk_\perp^2+2\hat{\alpha}_k\right)\left\langle|n_k|^2\right\rangle-2\hat{\alpha}_k\Re\langle n_k\phi_k^\star\rangle+2\omega_{\star e}\Im\langle n_k\phi_k^\star\rangle\\ &=2\Re\sum_{\vec{p}+\vec{q}=\vec{k}}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\Theta_{kpq}\left[a_p\left(k^2-q^2\right)\langle n_k^\star\phi_k\rangle\langle n_q\phi_q^\star\rangle\right.\\ &\left.+b_p\left(\langle n_q\phi_q^\star\rangle\left\langle|n_k|^2\right\rangle-\langle n_k^\star\phi_k\rangle\left\langle|n_q|^2\right\rangle\right)\right]\\ &+2\Re\sum_{\vec{p}+\vec{q}=\vec{k}}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\Theta_{kpq}\left[c_q\left(p^2-k^2\right)\langle n_k^\star\phi_k\rangle\left\langle|\phi_p|^2\right\rangle\right.\\ &\left.+d_q\left(\langle n_k^\star\phi_k\rangle\langle n_p^\star\phi_p\rangle-\left\langle|n_k|^2\right\rangle\left\langle|\phi_p|^2\right\rangle\right)\right]\end{aligned}$$

$$+\Re\sum_{\vec{p}+\vec{q}=\vec{k}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\Theta_{kpq}c_k^\star\left(q^2-p^2\right)\left(\left\langle n_q\phi_q^\star\right\rangle\left\langle|\phi_p|^2\right\rangle-\left\langle n_p\phi_p^\star\right\rangle\left\langle|\phi_q|^2\right\rangle\right)+\Re\sum_{\vec{p}+\vec{q}=\vec{k}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\Theta_{kpq}2d_k^\star\left(\left\langle|n_q|^2\right\rangle\left\langle|\phi_p|^2\right\rangle\left\langle n_q\phi_q^\star\right\rangle\left\langle n_p^\star\phi_p\right\rangle\right). \quad (39)$$

The first and second terms on the right hand side result from the coherent parts, $(\langle n_k^\star\delta\phi_p n_q\rangle-\langle n_k^\star\delta\phi_q n_p\rangle)$ and $(\langle n_k^\star\phi_p\delta n_q\rangle-\langle n_k^\star\phi_q\delta n_p\rangle)$ respectively, of the triad correlation on the right hand side of equation (18). The last two terms on the right hand side result from the incoherent part, $(\langle\delta n_k^\star\phi_p n_q\rangle-\langle\delta n_k^\star\phi_q n_p\rangle)$, of the triad correlation. It is easy to verify that in the strongly adiabatic limit $\hat{\alpha}\to\infty, a=b=c=d$ and $n_k=\phi_k$, which makes the transfer function (the entire right hand side) vanish. Equation (39) reveals that transfer of internal energy fluctuation is dominated by local interactions $(p\sim q)$.

In the purely adiabatic limit $\hat{\alpha}\to\infty$, the equations (35) and (39) reduce to the spectral equation for turbulence intensity of H–M system:

$$\frac{\partial}{\partial t}\left\langle|\phi_k|^2\right\rangle+2\eta_k\left\langle|\phi_k|^2\right\rangle=F_k. \quad (40)$$

Here the eddy damping rate is

$$\eta_k=-\Re\sum_{\vec{p}+\vec{q}=\vec{k}}M_{kpq}M_{pqk}\Theta_{kpq}\left\langle|\phi_q|^2\right\rangle \quad (41)$$

and the incoherent noise is

$$F_k=\Re\sum_{\vec{p}+\vec{q}=\vec{k}}M_{kpq}^2\Theta_{kpq}\left\langle|\phi_p|^2\right\rangle\left\langle|\phi_q|^2\right\rangle. \quad (42)$$

3.1. Induced diffusion of non-linear invariants by zonal modes

In this section, we calculate the effect of zonal modes on wavy scales. The dominance of nonlocal spectral transfer and the accumulation of energy in the natural repository of zonal modes suggest that the direct effect of zonal modes is of primary importance. This is calculated without further assumptions. In the following, we show how zonal modes induce diffusion of quadratic non-linear invariants in k_x-space.

3.1.1. Induced diffusion of spectral kinetic energy and internal energy at $\hat{\alpha}\neq\infty$.

In non-adiabatic $(\hat{\alpha}\neq\infty)$ case, the non-linear invariants are kinetic energy, internal energy, enstrophy, and cross-helicity. We show that the turbulence kinetic energy diffuses in k_x-space under the influence of zonal modes. This clarifies a key result of adiabatic theory. Assume that $\vec{q}$ is a zonal wave number. For convenience, we re-write the coherent and the noise terms on the right hand side of the spectral kinetic energy equation (35) as

$$T_{\phi k}^{(1)}\equiv\Re\sum_{\vec{p}+\vec{q}=\vec{k}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\left(q^2-p^2\right)\Theta_{kpq}2a_p\left(k^2-q^2\right)\times\left\langle|\phi_q|^2\right\rangle\left\langle|\phi_k|^2\right\rangle+\Re\sum_{\vec{p}+\vec{q}=\vec{k}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\left(q^2-p^2\right)\Theta_{kpq}a_k^\star\left(q^2-p^2\right)\times\left\langle|\phi_p|^2\right\rangle\left\langle|\phi_q|^2\right\rangle$$

$$T_{\phi k}^{(2)}\equiv\Re\sum_{\vec{p}+\vec{q}=\vec{k}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\left(q^2-p^2\right)\Theta_{kpq}2b_p\left(\left\langle|\phi_q|^2\right\rangle\langle n_k\phi_k^\star\rangle-\left\langle|\phi_k|^2\right\rangle\langle n_q^\star\phi_q\rangle\right)+\Re\sum_{\vec{p}+\vec{q}=\vec{k}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\left(q^2-p^2\right)\Theta_{kpq}b_k^\star\left(\left\langle|\phi_p|^2\right\rangle\langle n_q^\star\phi_q\rangle-\left\langle|\phi_q|^2\right\rangle\langle n_p^\star\phi_p\rangle\right).$$

Using the detailed balance equation (13) for the kinetic energy $E_k=\frac{1}{2}k^2\left\langle|\phi|_k^2\right\rangle$, $T_{\phi k}^{(1)}$ can be expressed as

$$T_{\phi k}^{(1)}=\Re\sum_{\vec{p}+\vec{q}=\vec{k}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\left(q^2-p^2\right)\times\left(k^2-q^2\right)E_q\Theta_{kpq}^E\left[a_p^E E_k-a_{-k}^E E_p\right] \quad (43)$$

where now $\Theta_{kpq}^E=\frac{8\Theta_{kpq}}{k_\perp^2p_\perp^2q_\perp^2}$ and $a_k^E=\frac{1}{2}k_\perp^2a_k$. Similarly, ignoring the zonal density potential correlation $\langle n_q^\star\phi_q\rangle$, $T_{\phi k}^{(2)}$ can be expressed as

$$T_{\phi k}^{(2)}=\Re\sum_{\vec{p}+\vec{q}=\vec{k}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\left(q^2-p^2\right)E_q\Theta_{kpq}^E\times\left[b_p^ER_{nk}E_k-b_{-k}^ER_{np}^\star E_p\right]. \quad (44)$$

Neglect of zonal correlation can be justified in case of evolution for $\alpha>1$. Here we focus in detail on the physics processes. Now, as shown in appendix A, expanding $T_{\phi k}^{(1)}$ and $T_{\phi k}^{(2)}$ around $\vec{p}=\vec{k}$ and retaining terms up to to $\mathcal{O}(q_x^4)$ yields the following,

$$T_{\phi k}^{(1)}=\frac{\partial}{\partial k_x}\left[\sum_q\frac{1}{2}k_y^2k^4q_x^4E_q\Theta_{kkq}^{Er}\left(a_k^{Er}\frac{\partial E_k}{\partial k_x}-\frac{\partial a_k^{Er}}{\partial k_x}E_k\right)\right]=\frac{\partial}{\partial k_x}\left[\sum_q4k_y^2\left(\frac{k}{k_\perp}\right)^4q_x^2E_q\Theta_{kkq}^{(r)}\left(a_k^{Er}\frac{\partial E_k}{\partial k_x}-\frac{\partial a_k^{Er}}{\partial k_x}E_k\right)\right] \quad (45)$$

and

$$T_{\phi k}^{(2)}=\frac{1}{k^2}\frac{\partial}{\partial k_x}\left[\sum_q\frac{1}{2}k_y^2k^4q_x^4E_q\Theta_{kkq}^{Er}\times\left(b_k^E\frac{\partial}{\partial k_x}\left(R_{nk}E_k\right)-\frac{\partial b_k^E}{\partial k_x}R_{nk}E_k\right)^{(r)}\right]=\frac{1}{k^2}\frac{\partial}{\partial k_x}\left[\sum_q2k_y^2\left(\frac{k}{k_\perp}\right)^4q_x^2E_q\Theta_{kkq}^{(r)}\right.$$

$$\times\left(b_k^E\frac{\partial}{\partial k_x}k^2\langle n_k\phi_k^\star\rangle-\frac{\partial b_k^E}{\partial k_x}k^2\langle n_k\phi_k^\star\rangle\right)^{(r)}\Bigg]. \quad (46)$$

Here $\Theta_{kkq}^{Er}=\Re\Theta_{kkq}^E$, $a_k^{Er}=\Re a_k^E$ and E_q refers to kinetic energy of large scales. Equation (45) shows that spectral turbulence kinetic energy E_k is convected and diffused in k-space by by large scale straining due to zonal shear kinetic energy $q_x^2E_q$. The sign of the convection speed depends on the sign of $\frac{\partial a_k^{Er}}{\partial k_x}$. It turns out that $\frac{\partial a_k^{Er}}{\partial k_x}<0$ hence, the sign of the convection speed is positive. $\Theta_{kkq}^{(r)}$ sets the strain coherence time. Equation (46) has structure of diffusion of spectral cross-correlation $k^2\langle n_k\phi_k^\star\rangle$. This shows that the diffusion of spectral kinetic energy is coupled to the diffusion of spectral cross-correlation. In fact, all non-linear invariants can be shown to diffuse in k-space by zonal mode scattering. Ignoring the zonal-cross correlation $\langle n_q^\star\phi_q\rangle$, it is convenient to split the internal energy transfer function into:

$$T_{nk}^{(1)}=2\Re\sum_{\vec p+\vec q=\vec k}(\hat z\cdot\vec p\times\vec q)^2\left\langle|\phi_q|^2\right\rangle\Theta_{kpq}\times\left[d_k^\star\left\langle|n_p|^2\right\rangle-d_p\left\langle|n_k|^2\right\rangle\right]$$

$$T_{nk}^{(2)}=2\Re\sum_{\vec p+\vec q=\vec k}(\hat z\cdot\vec p\times\vec q)^2\left\langle|\phi_q|^2\right\rangle\Theta_{kpq}\left[\left(q^2-k^2\right)c_p\langle n_k^\star\phi_k\rangle-\left(q^2-p^2\right)c_k^\star\langle n_p\phi_p^\star\rangle\right]$$

$$T_{nk}^{(3)}=2\Re\sum_{\vec p+\vec q=\vec k}(\hat z\cdot\vec p\times\vec q)^2\Theta_{kpq}\left[-b_p\langle n_k^\star\phi_k\rangle\left\langle|n_q|^2\right\rangle\right].$$

Now, as shown in appendix B, expanding $T_{nk}^{(1)}$ and $T_{nk}^{(2)}$ around $\vec p=\vec k$ and retaining terms up to to $\mathcal{O}(q_x^4)$ yields the following,

$$T_{nk}^{(1)}=\frac{\partial}{\partial k_x}\left[\sum_q k_y^2q_x^4\left\langle|\phi_q|^2\right\rangle\Theta_{kkq}^{(r)}\times\left(d_k^r\frac{\partial}{\partial k_x}\left\langle|n_k|^2\right\rangle-\frac{\partial d_k^r}{\partial k_x}\left\langle|n_k|^2\right\rangle\right)\right] \quad (47)$$

$$T_{nk}^{(2)}=\frac{\partial}{\partial k_x}\left[\sum_q k_y^2q_x^4\left\langle|\phi_q|^2\right\rangle\Theta_{kkq}^{(r)}\times\left(c_k^\star\frac{\partial}{\partial k_x}k^2\langle n_k\phi_k^\star\rangle-\frac{\partial c_k}{\partial k_x}k^2\langle n_k^\star\phi_k\rangle\right)^{(r)}\right]. \quad (48)$$

Here $q_x^4\left\langle|\phi_q|^2\right\rangle$ is energy associated with zonal velocity shear—i.e zonal vorticity. Equation (47) shows that spectral internal energy is convected and diffused in k-space by mean square zonal velocity shear. The sign of the convection speed depends on the sign of $\frac{\partial d_k^r}{\partial k_x}$. It can be checked that $\frac{\partial d_k^r}{\partial k_x}<0$, and hence, the sign if the convection speed is positive. Again, equation (48) has structure of diffusion of spectral cross-correlation $k^2\langle n_k\phi_k^\star\rangle$. This shows that the diffusion of spectral internal energy is coupled to the diffusion of spectral cross-helicity.

3.1.2. Induced diffusion of spectral total energy and enstrophy at $\hat\alpha=\infty$.

Let's define a generalized invariant $Q_k=\sigma_k^Q\left\langle|\phi_k|^2\right\rangle$, where $Q=(E,Z)$. Then the spectral equation for Q_k becomes:

$$\frac{\partial Q_k}{\partial t}=2\sum_{\vec p+\vec q=\vec k}M_k^QM_p^Q\Theta_{kpq}^QQ_qQ_k+\sum_{\vec p+\vec q=\vec k}\left|M_k^Q\right|^2\Theta_{kpq}^QQ_pQ_q \quad (49)$$

where $M_k^Q=\sigma_k^QM_{kpq}$ etc and $\Theta_{kpq}^Q=\Re\Theta_{kpq}/\sigma_k^Q\sigma_p^Q\sigma_q^Q$. Using the detailed balance equation (16) the non-linear transfer function can be reduced to

$$\begin{aligned}T_k&=2\sum_{\vec p+\vec q=\vec k}M_k^QM_p^Q\Theta_{kpq}^QQ_q\left(Q_k-Q_p\right)\\&=2\sum_{\vec q}k_y^2q_x^2\frac{\sigma_k^Q}{1+k^2}\left(q^2-p^2\right)\frac{\sigma_k^Q}{1+p^2}\\&\quad\times\left(k^2-q^2\right)\Theta_{kpq}^QQ_q\left(Q_k-Q_p\right).\end{aligned} \quad (50)$$

Assume that $\vec q$ is a zonal wave number. As shown in the appendix C, expanding around $\vec p=\vec k$ and retaining terms up to to $\mathcal{O}(q_x^4)$ yields the following,

$$T_k=\frac{\partial}{\partial k_x}\left[\sum_q k_y^2q_x^4Q_qk^4\left(\frac{\sigma_k^Q}{1+k^2}\right)^2\Theta_{kpq}^Q\frac{\partial Q_k}{\partial k_x}\right], \quad (51)$$

where Q_q can be either zonal kinetic energy or zonal enstrophy. This shows that in the pure adiabatic limit, all non-linear invariants are diffused in k_x-space by mean square zonal velocity shear.

3.2. Spectral evolution of zonal intensity

Here we calculate the spectral evolution equation of zonal kinetic energy—i.e. the energy of zonal flows. This analysis elucidates the fundamental mechanisms of zonal flow excitation, without further assumption. In particular, no explicit appeal to the adiabatic approximation is invoked. For zonal mode $k_y=k_\parallel=0$, hence the kinetic energy spectrum equation (17) for the zonal mode becomes

$$\left(\frac{\partial}{\partial t}+2\mu k_\perp^2\right)k_\perp^2\left\langle|\phi_k|^2\right\rangle=\Re\sum_{\vec k=\vec p+\vec q}\hat z\cdot\vec p\times\vec q\left(q^2-p^2\right)\langle\phi_k^\star\phi_p\phi_q\rangle. \quad (52)$$

Now the triplet correlations are approximated as

$$\langle\phi_k^\star\phi_p\phi_q\rangle=\langle\delta\phi_k^\star\phi_p\phi_q\rangle+\langle\phi_k^\star\delta\phi_p\phi_q\rangle+\langle\phi_k^\star\phi_p\delta\phi_q\rangle. \quad (53)$$

The zonal perturbation $\delta\phi_k$ is driven by the beat interaction between modes $\vec{p}$ and $\vec{q}$:

$$\left(\frac{\partial}{\partial t}+\eta_k\right)k_x^2\delta\phi_k = S_{1k} \tag{54}$$

where S_{1k} is same as given in equation (23). The solution of the zonal beat mode equation (54) is

$$\delta\phi_k = \frac{1}{k_x^2}\int_{-\infty}^{t} dt' e^{-\eta_k(t-t')} S_{1k}(t'). \tag{55}$$

Then the incoherent part of the triplet correlation becomes

$$\begin{aligned}\langle\delta\phi_k^\star\phi_p\phi_q\rangle &= \frac{1}{k_x^2}\int_{-\infty}^{t} dt' e^{-\eta_k(t-t')}\langle S_{1k}^\star(t')\phi_p(t)\phi_q(t)\rangle\\ &= \Theta_{kpq}\frac{1}{k_x^2}\hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)\left\langle|\phi_p|^2\right\rangle\left\langle|\phi_q|^2\right\rangle\end{aligned}$$

where the triad interaction time is

$$\Theta_{kpq} = \frac{1}{i\left(\omega_p+\omega_q\right)+\eta_k+\eta_p+\eta_q}. \tag{56}$$

It is easy to see that the coherent part of the triplet correlation is same as obtained in equation (34), with the expression for Θ_{kpq} given by equation (56). Hence the zonal spectral intensity equation becomes:

$$\left(\frac{\partial}{\partial t}+2\mu k_x^2\right)\left\langle|\phi_k|^2\right\rangle+2\eta_{1k}^{(r)}\left\langle|\phi_k|^2\right\rangle+\Re\left[2\eta_{2k}\langle n_k\phi_k^\star\rangle\right] = F_{\phi k}. \tag{57}$$

In the above equation, the second term on the left proportional to zonal intensity represents non-linear damping of zonal flow with the damping rate

$$\begin{aligned}\eta_{1k}^{(r)} = &-\Re\sum_{\vec{k}=\vec{p}+\vec{q}}\frac{1}{k_x^2}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\left(q^2-p^2\right)\Theta_{kpq}\\ &\times\left[a_p\left(k^2-q^2\right)-b_p\frac{\langle n_q^\star\phi_q\rangle}{\left\langle|\phi_q|^2\right\rangle}\right]\left\langle|\phi_q|^2\right\rangle.\end{aligned}$$

The second term on the left hand side shows coupling to zonal cross correlation $\langle n_k\phi_k^\star\rangle$ with the cross coupling coefficient given by

$$\eta_{2k} = -\sum_{\vec{k}=\vec{p}+\vec{q}}\frac{1}{k_x^2}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\left(q^2-p^2\right)\Theta_{kpq}b_p\left\langle|\phi_q|^2\right\rangle.$$

This is a novel effect! Finally, the term on the right hand side is the zonal non-linear noise:

$$\begin{aligned}F_{\phi k} = &\Re\sum_{\vec{k}=\vec{p}+\vec{q}}\frac{1}{(k_x^2)^2}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\left(q^2-p^2\right)^2\Theta_{kpq}\\ &\times\left\langle|\phi_p|^2\right\rangle\left\langle|\phi_q|^2\right\rangle.\end{aligned} \tag{58}$$

Note that the zonal noise term here is exactly the same as the zonal noise term for the H–M equation and is positive definite. It is determined by the advection of vorticity. However, the eddy damping term is different from the corresponding H–M case due to non-adiabaticity of electrons. The electron non-adiabaticity parameter enters through the coupling parameters a_p, b_p, the turbulent density potential correlation $\langle n_q^\star\phi_q\rangle$ and the triad interaction time Θ_{kpq}. In the following we use the linear density-potential response relation for wavy modes to simplify $\langle n_q^\star\phi_q\rangle$ correlations, so that $\langle n_q^\star\phi_q\rangle = R_{nq}^\star I_q$. This is defensible only for the $\alpha > 1$ regime, where density and potential fluctuations are strongly correlated and the density response is laminar. We leave the zonal density potential correlation for later discussion—i.e. $\langle n_k\phi_k^\star\rangle$ appears explicitly in the theory. Using $k^2 \ll q^2$ and expanding the triad interaction time Θ_{kpq} around $\vec{p} = -\vec{q}$

$$\Theta_{kpq}^{(r)} \approx \Theta_{k,-q,q}^{(r)}+\vec{k}\cdot\frac{\partial\Theta_{kpq}^{(r)}}{\partial\vec{p}}|_{\vec{p}=-\vec{q}} = \Theta_{k,-q,q}^{(r)}-\frac{\vec{k}}{2}\cdot\frac{\partial\Theta_{k,-q,q}^{(r)}}{\partial\vec{q}} \tag{59}$$

the non-linear damping rate is seen to be:

$$\begin{aligned}\eta_{1k}^{(r)} &= -\sum_{\vec{q}}k_r^2q_y^2\left(\Theta_{k,-q,q}^{(r)}+q_x\frac{\partial\Theta_{k,-q,q}^{(r)}}{\partial q_x}\right)\\ &\quad\times\left(a_{-q}q^2+b_{-q}R_{nq}^\star\right)^{(r)}I_q\\ &= -\sum_{\vec{q}}\frac{\partial}{\partial q_x}\left[k_x^2q_y^2q_x\Theta_{k,-q,q}^{(r)}\left(a_{-q}q^2+b_{-q}R_{nq}^\star\right)^{(r)}I_q\right]\\ &\quad+\sum_{\vec{q}}k_x^2q_y^2\Theta_{k,-q,q}^{(r)}q_x\frac{\partial}{\partial q_x}\left[\left(a_{-q}q^2+b_{-q}R_{nq}^\star\right)^{(r)}I_q\right]\end{aligned} \tag{60}$$

where the first (surface) term vanishes. So the general expression for the non-linear damping rate becomes

$$\eta_{1k}^{(r)} = \sum_{\vec{q}}k_x^2q_y^2\Theta_{k,-q,q}^{(r)}q_x\frac{\partial}{\partial q_x}\left[\left(a_{-q}q^2+b_{-q}R_{nq}^\star\right)^{(r)}I_q\right]. \tag{61}$$

Since, $\Re\left[\eta_{2k}\langle n_k\phi_k^\star\rangle\right] = \eta_{2k}^{(r)}\langle n_k\phi_k^\star\rangle^{(r)}-\eta_{2k}^{(i)}\langle n_k\phi_k^\star\rangle^{(i)}$, one needs to evaluate both real and imaginary parts of the cross-coefficient η_{2k}^{zonal}. Using the above expansion procedure, the real part of η_{2k} becomes

$$\eta_{2k}^{(r)} = -\sum_{\vec{q}}k_x^2q_y^2\Theta_{k,-q,q}^{(r)}q_x\frac{\partial}{\partial q_x}\left[b_{-q}^{(r)}I_q\right] \tag{62}$$

and the imaginary part becomes

$$\eta_{2k}^{(i)} = -\sum_{\vec{q}}k_x^2q_y^2\Theta_{k,-q,q}^{(r)}q_x\frac{\partial}{\partial q_x}\left[b_{-q}^{(i)}I_q\right] = 0. \tag{63}$$

Note that $\eta_{2k}^{(i)} = 0$ due to the q_y-symmetry of $b_q^{(i)}$—i.e. it is odd in q_y. This means that only $\langle n_k\phi_k^\star\rangle^{(r)}$—i.e. the real part of the zonal cross-spectrum affects the evolution of zonal intensity. The zonal noise term can be reduced to

$$F_{\phi k} = \sum_q 4q_y^2 q_x^2 \Theta^{(r)}_{k,-q,q} I_{-q}(t) I_q(t) + \mathcal{O}(k_x^2/q_x^2)$$
$$\approx 4\sum_q \Pi_q^2 \Theta^{(r)}_{k,-q,q} \tag{64}$$

where $\Pi_q = q_y q_x I_q$ is spectral form of Reynolds stress.

3.2.1. Adiabatic regime $\omega_q < \hat{\alpha}_q$. The linear density potential response function, in the weakly adiabatic regime can be reduced to

$$R_{nq} = \left(1 - i\frac{\omega_{\star e}}{\hat{\alpha}_q}\right)\left(1 - i\frac{\omega}{\hat{\alpha}_q}\right)^{-1}$$
$$= 1 + \frac{q_\perp^4}{1+q_\perp^2}\frac{1}{\alpha_q^2} - \frac{iq_\perp^2}{\alpha_q} + \mathcal{O}\left(\frac{1}{\alpha_q^3}\right). \tag{65}$$

The coupling parameters in the adiabatic regime become

$$a_q = \left(1 - \frac{i}{\alpha_q} + \frac{1}{1+q^2}\frac{q_\perp^2}{\alpha_q^2}\right) b_q \tag{66}$$

$$b_q = \frac{1}{1+q^2}\left(1 + i\frac{2}{1+q^2}\frac{q^2}{\alpha_q}\right) + \mathcal{O}\left(\frac{1}{\alpha_q^2}\right). \tag{67}$$

Using the expression for R_q in the adiabatic regime, the non-linear zonal damping rate becomes

$$\eta^{(r)}_{1k} = \sum_q k_x^2 q_y^2 \Theta^{(r)}_{k,-q,q} q_x \frac{\partial}{\partial q_x}\left[\left(1 - \frac{2q_\perp^4}{(1+q_\perp^2)^2}\frac{1}{\alpha_q^2}\right) I_q\right]. \tag{68}$$

This shows that the non-linear damping of zonal flow is negative when the turbulence intensity spectra satisfies $\frac{\partial I_q}{\partial q_r} < 0$, which is usually the case. In this case, negative viscosity results i.e. $\eta^{(r)}_{1k} < 0$ and $\sim k_x^2$, symptomatic of transfer to large scales by negative viscosity. The total growth $\mathcal{G}_k$ of zonal flows is determined by $\eta^{(r)}_{1k}$ and the linear damping μk_x^2, so, $\mathcal{G}_k = -\eta_{1k} - \mu k_x^2$. $\mathcal{G}_k$ defines a critical spectral slope for marginality to modulational instability. It is also clear that the zonal growth rate is maximum for the strongly adiabatic regime, when $\alpha_q \to \infty$. This suggests that non-adiabatic density fluctuations inhibit the inverse transfer of energy to zonal flows.

The cross-coefficient $\eta^{(r)}_{2k}$ is independent of α since $b^{(r)}_q$ (from equation (67)) is independent of α. Hence, $\eta^{(r)}_{2k}$ is always positive for negative spectral slope. This means that the zonal cross correlation can cause either forward or inverse transfer of energy, depending on the sign of the cross-correlation $\langle n_k \phi_k^\star \rangle$, i.e. the relative phase between zonal density and potential.

3.3. Spectral evolution of density corrugations

Here we calculate the generation of zonal density perturbations by non-linear interaction of wavy modes. Note that the mechanisms for zonal density (which is intrinsically non-adiabatic) generation can not be presumed to be the same as for zonal flows. For the zonal mode $k_y = k_\parallel = 0$, hence the spectral equation (18) for zonal internal energy/density corrugations becomes

$$\left(\frac{\partial}{\partial t} + 2D_n k^2\right)\left\langle |n_k|^2 \right\rangle = \Re \sum_{\vec{p}+\vec{q}=\vec{k}} \hat{z}\cdot\vec{p}\times\vec{q}\,(\langle n_k^\star \phi_p n_q\rangle$$
$$- \langle n_k^\star \phi_q n_p\rangle). \tag{69}$$

Now using the procedure outlined for the zonal flow energy derivation, it is straightforward to arrive at the following equation for the density corrugations intensity,

$$\left(\frac{\partial}{\partial t} + 2D_n k^2\right)\left\langle |n_k|^2 \right\rangle + 2\zeta^{(r)}_{1k}\left\langle |n_k|^2 \right\rangle + \Re\left[2\zeta_{2k}\langle n_k^\star \phi_k\rangle\right] = F_{nk}. \tag{70}$$

Here, the corrugations damping rate due to turbulent mixing is

$$\zeta^{(r)}_{1k} = \Re \sum_{\vec{k}=\vec{p}+\vec{q}} \Theta_{kpq}(\hat{z}\cdot\vec{p}\times\vec{q})^2\left[d_p\langle \phi_q \phi_q^\star\rangle - b_p \langle \phi_q^\star n_q\rangle\right],$$

the coefficient of coupling to zonal cross correlation is

$$\zeta_{2k} = \sum_{\vec{k}=\vec{p}+\vec{q}} \Theta_{kpq}(\hat{z}\cdot\vec{p}\times\vec{q})^2\left[a_p(q^2 - k^2)\langle \phi_q^\star n_q\rangle\right.$$
$$\left. + b_p\langle n_q^\star n_q\rangle + c_q(k^2 - p^2)\langle \phi_p \phi_p^\star\rangle - d_q\langle \phi_p n_p^\star\rangle\right]$$

and the noise is

$$F_{nk} = 2\Re \sum_{\vec{k}=\vec{p}+\vec{q}} \Theta_{kpq}(\hat{z}\cdot\vec{p}\times\vec{q})^2\left[\langle \phi_p^\star \phi_p\rangle\langle n_q^\star n_q\rangle\right.$$
$$\left. - \langle \phi_q^\star n_q\rangle\langle n_p^\star \phi_p\rangle\right].$$

The corrugations damping and the zonal cross-correlation result from the coherent parts, $(\langle n_k^\star \delta\phi_p n_q\rangle - \langle n_k^\star \delta\phi_q n_p\rangle)$ and $(\langle n_k^\star \phi_p \delta n_q\rangle - \langle n_k^\star \phi_q \delta n_p\rangle)$, of the triad correlation on the right hand side of equation (69). The noise term $F_{n\phi}$ results from the incoherent part, $(\langle \delta n_k^\star \phi_p n_q\rangle - \langle \delta n_k^\star \phi_q n_p\rangle)$, of the triad correlation. Clearly, like zonal flow energy, the evolution of corrugation intensity is dynamically coupled to the zonal cross-correlation spectrum. Now, assuming linear density potential response for the turbulent density potential correlations i.e. $\langle \phi_q^\star n_q\rangle = R_{nq} I_q$ and using the fact that $k^2 \ll q^2$ and expanding around $\vec{p} = -\vec{q}$ yields the damping rate:

$$\zeta^{(r)}_{1k} = \Re \sum_{\vec{q}} \Theta_{k,-q,q} k_x^2 q_y^2 \left[d_{-q} - b_{-q} R_{nq}\right] I_q.$$

Similarly, the cross-coefficient becomes

$$\zeta_{2k} = \sum_{\vec{q}} \Theta_{k,-q,q} k_x^2 q_y^2 \left[q^2\left(a_q^\star R_q - c_q\right) + b_q^\star |R_{nq}|^2 - d_q R_{nq}\right] I_q.$$

In general, this is a complex quantity. But, using the q_y symmetry properties of the coefficients a_q, b_q, c_q, d_q and R_q, it straight-forward to show that imaginary part of ζ_{2k} vanish i.e. $\zeta_{2k}^{(i)}=0$. Hence $\Re[2\zeta_{2k}\langle n_k^\star\phi_k\rangle]=\zeta_{2k}^{(r)}\langle n_k^\star\phi_k\rangle^{(r)}$. That is only the real part of the zonal cross-spectrum couples to the corrugation intensity evolution. Finally, the noise term reduces to

$$\begin{aligned}F_{nk}&=2\Re\sum_{\vec q}\Theta_{k,-q,q}k_x^2q_y^2\left[|R_{nq}|^2-R_{nq}^2\right]I_q^2\\&=4\sum_{\vec q}\Theta_{k,-q,q}^{(r)}k_x^2q_y^2\left(R_{nq}^i\right)^2I_q^2.\end{aligned}$$

A negative damping rate would mean modulational growth of density corrugations. We shall see, however, that this is not the case. The noise term is always positive definite. It is important to note that, in contrast to the case of zonal flows, both modulational growth and the corrugation noise are independent of the spectral slope. In contrast, the modulational growth of zonal flow requires a negative spectral slope of kinetic energy.

3.3.1. Adiabatic regime $\omega_q<q_\parallel^2\chi_e$.

The coupling parameters in the adiabatic regime become

$$b_q=\frac{1}{1+q^2}\left(1+i\frac{2}{1+q^2}\frac{q^2}{\alpha_q}\right)+\mathcal{O}\left(\frac{1}{\alpha_q^2}\right)$$

$$\begin{aligned}c_q&=\left(1-i\frac{q^2}{\alpha_q}+\frac{q_\perp^4}{1+q_\perp^2}\frac{1}{\alpha_q^2}\right)a_q\\&=\frac{1}{1+q^2}\left(1-\frac{i}{\alpha_q}\frac{2+\left(1+q^2\right)^2}{1+q^2}-\frac{2q^2}{\alpha_q^2}\right)+\mathcal{O}\left(\frac{1}{\alpha_q^3}\right)\end{aligned}$$

$$d_q=\left(1+\frac{q_\perp^4}{1+q_\perp^2}\frac{1}{\alpha_q^2}-i\frac{q^2}{\alpha_q}\right)b_q.$$

Using the expression for R_q in the adiabatic regime one finds the non-linear density corrugation damping rate:

$$\zeta_{1k}^{(r)}=\sum_{\vec q}\Theta_{k,-q,q}^{(r)}k_x^2q_y^2\frac{4q^4}{\left(1+q^2\right)^2}\frac{1}{\alpha_q^2}I_q. \tag{71}$$

Note that sign of the damping rate ζ_k is positive definite in the adiabatic regime! This means that the zonal density corrugations are modulationally stable—indeed diffusively damped—for $\alpha>1$. In, contrast to the case of zonal flows, distant interaction of small and large scales does not generate density corrugations. After some algebraic manipulations, one can show that the cross-coefficient $\zeta_{2k}^{(r)}$ becomes

$$\zeta_{2k}^{(r)}=\sum_{\vec q}\Theta_{k,-q,q}^{(r)}k_x^2q_y^2\frac{\left(1+2q^2\right)}{\left(1+q^2\right)^2}\frac{q^4}{\alpha_q^2}I_q.$$

Similarly, the the noise term becomes

$$F_{nk}=4\sum_{\vec q}\Theta_{k,-q,q}^{(r)}k_x^2q_y^2\frac{q_\perp^4}{\alpha_q^2}I_q^2. \tag{72}$$

Note that the modulational growth $\zeta_{1k}^{(r)}$, the cross-coefficient $\zeta_{2k}^{(r)}$ and the noise F_{nk}—all scale as $\frac{1}{\alpha_q^2}$, and are positive definite. This means the density corrugations get weaker as α increases,—i.e. as the response become more adiabatic.

3.4. Spectral evolution of zonal cross-correlations

It is interesting to note that several of the results of this section depend sensitively upon the cross-correlation $\langle n_k^\star\phi_k\rangle$. The impact of cross-correlation on spectral transfer process has long been appreciated in the context of waves and transport [59]. This was discussed above. However, the significance of zonal cross-correlation has not been appreciated, and is discussed here for the first time. While the cross correlation for the wavy component can be simplified (for $\alpha>1$) by using the linear response, this is not valid for the zonal modes. Moreover the accumulation of energy in the zonal modes suggests that zonal cross correlation merits special attention. Finally, we note that the cross-spectrum encodes information concerning the relative phasing of zonal shears and density corrugations. Thus it is central to the description of staircases, and other spatial patterns [5, 6]. In staircases, zonal density and potential self-organize in a quasi-periodic pattern, and thus are spatially correlated. Here follows an approach to calculate the zonal correlation. Multiplying the zonal density equation by zonal vorticity (i.e. zonal shear) and multiplying the zonal vorticity equation by zonal density and adding the resulting equations yields

$$\frac{\partial}{\partial t}\bar n\nabla_x^2\bar\phi-\mu\bar n\nabla_x^4\bar\phi-D_n\nabla_x^2\bar\phi\nabla_x^2\bar n=-\nabla_x^2\bar\phi\nabla_x\Gamma_{nx}-\bar n\nabla_x^2\Pi_{xy} \tag{73}$$

where $\Gamma_{nx}=\overline{\tilde v_x\tilde n}$ is radial particle flux and $\Pi_{xy}=\overline{\tilde v_x\tilde v_y}$ is the Reynolds stress. Now considering $\langle\rangle\equiv\int dx/L$, the equation for the zonal correlation becomes

$$\begin{aligned}&\frac{\partial}{\partial t}\left\langle\bar n\nabla_x^2\bar\phi\right\rangle-\left(\mu+D_n\right)\left\langle\nabla_x^2\bar n\nabla_x^2\bar\phi\right\rangle\\&\quad=\left\langle\Gamma_{nx}\nabla_x^3\bar\phi\right\rangle+\left\langle\nabla_x\Pi_{xy}\nabla_x\bar n\right\rangle.\end{aligned} \tag{74}$$

This shows that the zonal correlation is determined by the correlations of profiles and fluxes. The first term on the right hand side is the correlation of the radial particle flux with zonal vorticity gradient and the second term is correlation of Reynolds force (vorticity flux) with the zonal density gradient. Thus zonal correlations are relevant to spatial structure of the profiles. Setting $k_y=k_\parallel=0$ in equation (19), the evolution equation for zonal cross-correlation spectrum becomes

$$\left(\frac{\partial}{\partial t}+\left(\mu+D_n\right)k_x^2\right)\left\langle n_k\phi_k^\star\right\rangle=\sum_{\vec k=\vec p+\vec q}\hat z\cdot\vec p\times\vec q$$

$$\times\left[\frac{(q^2-p^2)}{k_x^2}\left\langle n_k\phi_p^\star\phi_q^\star\right\rangle+\left\langle\phi_k^\star\phi_p n_q\right\rangle-\left\langle\phi_k^\star\phi_q n_p\right\rangle\right]. \quad (75)$$

Now the triplet correlations are calculated as outlined in the previous sub-sections and details are provided in appendix D. It is straight forward to arrive at the following evolution equation for the real part of zonal correlation $\Re\langle n_k\phi_k^\star\rangle$.

$$\left(\frac{\partial}{\partial t}+(\mu+D_n)k_x^2\right)\Re\langle n_k\phi_k^\star\rangle+\Re[2\xi_{1k}\langle n_k\phi_k^\star\rangle] + 2\xi_{2k}^{(r)}\left\langle|n_k|^2\right\rangle+2\xi_{3k}^{(r)}\left\langle|\phi_k|^2\right\rangle=F_{n\phi k} \quad (76)$$

where the terms on the left hand side result from the coherent part of the triplet correlations and that on the right hand side is the incoherent noise term. Here

$$\xi_{1k}=\sum_{\vec{k}=\vec{p}+\vec{q}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\Theta_{kpq}^\star\frac{(q^2-p^2)}{k_x^2}\left[a_p^\star\left(q^2-k^2\right)\left\langle|\phi_q|^2\right\rangle+b_p^\star\left\langle n_q\phi_q^\star\right\rangle\right]+\sum_{\vec{k}=\vec{p}+\vec{q}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\Theta_{kpq}\left[d_q\left\langle|\phi_p|^2\right\rangle-b_p\left\langle n_q\phi_q^\star\right\rangle\right]=\eta_{1k}+\zeta_{1k}$$

so that its real part

$$\xi_{1k}^{(r)}=\eta_{1k}^{(r)}+\zeta_{1k}^{(r)}$$

is the sum of the zonal flow intensity and corrugation intensity damping rates. Expanding the imaginary part of ξ_{1k} about $\vec{p}=-\vec{q}$ and using the q_y-symmetry properties of the coefficients a_q, b_q, d_q and R_q shows that $\xi_{1k}^{(i)}=0$. This implies that $\Re[2\xi_{1k}\langle n_k\phi_k^\star\rangle]=2\xi_{1k}^{(r)}\Re\langle n_k\phi_k^\star\rangle$. The coefficient of coupling to corrugations intensity is

$$\xi_{2k}=-\sum_{\vec{k}=\vec{p}+\vec{q}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\Theta_{k,p,q}^\star\frac{(q^2-p^2)}{k_x^2}b_p^\star\left\langle|\phi_q|^2\right\rangle=\eta_{2k}$$

so that $\xi_{2k}^{(r)}=\eta_{2k}^{(r)}$, and the coefficient of coupling to zonal flow intensity is

$$\xi_{3k}=-\sum_{\vec{k}=\vec{p}+\vec{q}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\Theta_{kpq}\left[a_p\left(k^2-q^2\right)\left\langle\phi_q^\star n_q\right\rangle-b_p\left\langle|n_q|^2\right\rangle+c_q\left(p^2-k^2\right)\left\langle|\phi_p|^2\right\rangle+d_q\left\langle\phi_p n_p^\star\right\rangle\right]=\zeta_{2k}$$

so that $\xi_{3k}^{(r)}=\zeta_{2k}^{(r)}$. Finally, the noise term is

$$F_{n\phi k}=\Re\sum_{\vec{k}=\vec{p}+\vec{q}}(\hat{z}\cdot\vec{p}\times\vec{q})^2\frac{1}{k_x^2}\left(q^2-p^2\right)\left[\Theta_{kpq}^\star+\Theta_{kpq}\right]\times\left[\left\langle|\phi_p|^2\right\rangle\left\langle n_q\phi_q^\star\right\rangle-\left\langle|\phi_q|^2\right\rangle\left\langle n_p\phi_p^\star\right\rangle\right].$$

Expanding about $\vec{p}=-\vec{q}$, and using the linear density potential relation for the wavy mode $\langle n_q\phi_q^\star\rangle=R_qI_q$, it is obvious to see that $F_{n\phi k}=0$. Hence the evolution equation for $\Re\langle n_k\phi_k^\star\rangle$ becomes

$$\left(\frac{\partial}{\partial t}+(\mu+D_n)k_x^2\right)\Re\langle n_k\phi_k^\star\rangle+2\xi_{1k}^{(r)}\Re\langle n_k\phi_k^\star\rangle + 2\xi_{2k}^{(r)}\left\langle|n_k|^2\right\rangle+2\xi_{3k}^{(r)}\left\langle|\phi_k|^2\right\rangle=0. \quad (77)$$

In steady state,

$$\Re\langle n_k\phi_k^\star\rangle=\frac{2\xi_{2k}^{(r)}\left\langle|n_k|^2\right\rangle+2\xi_{3k}^{(r)}\left\langle|\phi_k|^2\right\rangle}{-(\mu+D_n)k_x^2-2\xi_{1k}^{(r)}}. \quad (78)$$

The sign of the real of the zonal cross correlation spectrum is determined by the sign of $-(\mu+D_n)k_x^2-2\xi_{1k}^{(r)}$, where $\xi_{1k}^{(r)}$ is the sum of the non-linear damping rates of the zonal intensity and density corrugation $\xi_{1k}^{(r)}=\eta_{1k}^{(r)}+\zeta_{1k}^{(r)}$. Note that zonal flow non-linear damping rate is negative while the non-linear damping rate for density corrugation is positive. This implies that the zonal cross correlation is positive when the $-\eta_{1k}^{(r)}-\zeta_{1k}^{(r)}>(\mu+D_n)k_x^2/2$ i.e. when the modulational growth of zonal flow intensity exceeds the non-linear damping rate of density corrugation by $(\mu+D_n)k_x^2/2$. Otherwise, the sign of $\Re\langle n_k\phi_k^\star\rangle$ is negative. A positive value of $\Re\langle n_k\phi_k^\star\rangle$ suggests that zonal density and zonal potential tend to align. A negative value of $\Re\langle n_k\phi_k^\star\rangle$ suggests that zonal density and zonal potential are anti-correlated. zonal correlation suggests that corrugations and shears tend to align. A negative value suggests that corrugations and shears are anti-correlated.

Note that this is a spectral correlation. Multiplying by $-k_x^2$ and summing over all $\vec{k}$ yields the correlation of zonal density and zonal vorticity

$$\left\langle n\nabla_x^2\phi\right\rangle=\sum_{k_x}-\Re\left\langle n_kk_x^2\phi_k^\star\right\rangle=\sum_{k_x}-k_x^2\frac{2\xi_{2k}^{(r)}\left\langle|n_k|^2\right\rangle+2\xi_{3k}^{(r)}\left\langle|\phi_k|^2\right\rangle}{-(\mu+D_n)k_x^2-2\xi_{1k}^{(r)}}. \quad (79)$$

To obtain correlation of density gradient and vorticity

$$\left\langle\nabla_xn\nabla_x^2\phi\right\rangle=\sum_{k_x}-\Re\left\langle ik_xn_kk_x^2\phi_k^\star\right\rangle=\sum_{k_x}k_x^3\Im\langle n_k\phi_k^\star\rangle \quad (80)$$

one has to obtain imaginary of the correlation spectra $\Im\langle n_k\phi_k^\star\rangle$. This must be obtained from equation (A24) in the appendix. From equation (A24) one can arrive at the following equation for $\Im\langle n_k\phi_k^\star\rangle$:

$$\left(\frac{\partial}{\partial t}+(\mu+D_n)k_x^2\right)\Im\langle n_k\phi_k^\star\rangle+2\xi_{1k}^{(r)}\Im\langle n_k\phi_k^\star\rangle + 2\xi_{1k}^{(i)}\Re\langle n_k\phi_k^\star\rangle+2\xi_{2k}^{(i)}\left\langle|n_k|^2\right\rangle+2\xi_{3k}^{(i)}\left\langle|\phi_k|^2\right\rangle=F_{n\phi k}^{(i)}. \quad (81)$$

Now it is straight forward to to show that the coupling coefficients $\xi_{1k}^{(i)}$, $\xi_{2k}^{(i)}$ and $\xi_{3k}^{(i)}$ and the imaginary part of the zonal cross correlation noise i.e. $F_{n\phi k}^{(i)}$, all vanish for a small k/q

expansion due to q_y—symmetry properties of the coupling coefficients $a^{(i)}$, $b^{(i)}$, $c^{(i)}$, $d^{(i)}$ and the imaginary of the response function $R_n^{(i)}$ i.e. all are odd in q_y. As a concrete example, we show how noise term vanishes in the following. The imaginary noise term $F^{(i)}_{n\phi k}$ is given by

$$F^{(i)}_{n\phi k} = \Im \sum_{\vec{k}=\vec{p}+\vec{q}} (\hat{z}\cdot\vec{p}\times\vec{q})^2 \frac{1}{k_x^2}\left(q^2-p^2\right)\left[\Theta^\star_{kpq}+\Theta_{kpq}\right] \times \left[\left\langle|\phi_p|^2\right\rangle\langle n_q\phi_q^\star\rangle - \left\langle|\phi_q|^2\right\rangle\langle n_p\phi_p^\star\rangle\right].$$

Now expanding about $\vec{p}=-\vec{q}$ and using $q^2-p^2=-k^2+2k_xq_x$ and $\Theta^{(r)}_{kpq}\approx\Theta^{(r)}_{k,-q,q}-\frac{\vec{k}}{2}\cdot\frac{\partial\Theta^{(r)}_{k,-q,q}}{\partial\vec{q}}$ one obtains

$$\begin{aligned} F^{(i)}_{n\phi k} &= -4\sum k_x^2q_y^2\left(\Theta^{(r)}_{k,-q,q}+q_x\frac{\partial\Theta^{(r)}_{k,-q,q}}{\partial q_x}\right) \times\left(I_qR^{(i)}_{nq}+\mathcal{O}(k_x)\right)\\ &= 4\sum k_x^2q_y^2\Theta^{(r)}_{k,-q,q}q_x\frac{\partial}{\partial q_x}\left(I_qR^{(i)}_{nq}+\mathcal{O}(k_x/q_x)\right). \end{aligned}$$

Note that $R^{(i)}_{nq}$ is odd in q_y and hence $F_{n\phi k}$ vanishes to leading order. Similarly, one can show that coupling coefficients $\xi^{(i)}_{1k}$, $\xi^{(i)}_{2k}$ and $\xi^{(i)}_{3k}$ vanish to leading order. Note that the imaginary of the triad response time for $\vec{p}=-\vec{q}$ is always zero i.e. $\Theta^{(i)}_{k,-q,q}=0$ and hence does not play any role in setting up the coupling coefficients in a low k expansion. Hence the equation for the imaginary cross correlation becomes

$$\left(\frac{\partial}{\partial t}+(\mu+D_n)k_x^2\right)\Im\langle n_k\phi_k^\star\rangle+2\xi^{(r)}_{1k}\Im\langle n_k\phi_k^\star\rangle=0. \quad (82)$$

Note that this equation is decoupled from the other three equations -zonal intensity equation (57), density corrugation equation (70) and the real of zonal cross correlation equation (76). The equation (82) can have either an exponentially growing or exponentially decaying solution depending on the sign of $(\mu+D_n)k_x^2+2\xi^{(r)}_{1k}$. Since this equation has no steady state solution, a physically constrained solution will be an exponentially decaying solution which vanish asymptotically i.e. $\Im\langle n_k\phi_k^\star\rangle=0$. This constrains $(\mu+D_n)k_x^2+2\xi^{(r)}_{1k}>0$.

3.4.1. Connection with observations. This implies that in DW turbulence the zonal density and potential cross-correlation is negative i.e. $\langle\overline{n}\overline{\phi}\rangle<0$. The zonal density and vorticity cross-correlation is then positive i.e. $\langle\overline{n}\nabla_x^2\overline{\phi}\rangle>0$. However the zonal density gradient and zonal vorticity cross-correlation should vanish as $\Im\langle n_k\phi_k^\star\rangle=0$ to leading order, i.e. $\langle\nabla_x\overline{n}\nabla_x^2\overline{\phi}\rangle=0$. The density gradient and vorticity gradient cross-correlation becomes positive i.e. $\langle\nabla_x\overline{n}\nabla_x^3\overline{\phi}\rangle>0$. This means that the *zonal density jumps are co-located with the zonal vorticity jumps*. Finally, another correlation of interest could be between zonal density gradient and zonal flow $\langle-\nabla_x\overline{n}\nabla_x\overline{\phi}\rangle$. This can be obtained as $\langle-\nabla_x\overline{n}\nabla_x\overline{\phi}\rangle=\sum_{k_x}-k_x^2\Re\langle n_k\phi_k^\star\rangle>0$. This means *density gradient peaks are co-located with the zonal flow peaks*. These results align with the observations of staircase features in the GYSELA simulations [6], with the caution that they studied temperature profile corrugation rather than density. For better comparison with simulations and experiments, temperature corrugation dynamics should also be investigated. This can be achieved through spectral closure theory of ITG turbulence. This clearly is a subject for a separate study.

3.4.2. A quasilinear alternative. The correlations $\langle\Gamma_{nx}\nabla_x^3\overline{\phi}\rangle$ and $\langle\nabla_x\Pi_{xy}\nabla_x\overline{n}\rangle$ can be obtained by quasilinear calculations as follows. Quasilinear expression for particle flux is obtained as

$$\Gamma_{nx}=\sum_k -k_yR^{(i)}_{nk}|\phi_k|^2$$

where the imaginary part of the density-potential response function is

$$R^{(i)}_{nk}=-\frac{\hat{\alpha}(\omega_{\star e}-\omega_r)+\gamma\omega_{\star e}}{|\omega+i\hat{\alpha}|^2}.$$

The dispersion relation with mean/zonal vorticity gradient $\nabla_x^3\overline{\phi}$ can be obtained as

$$k_\perp^2\omega_k^2+\omega_k\left\{i\hat{\alpha}\left(1+k_\perp^2\right)-k_y\nabla_x^3\overline{\phi}\right\}-i\hat{\alpha}\left\{k_y\nabla_x^3\overline{\phi}+\omega_{\star e}\right\}=0. \quad (83)$$

The the expression for real frequency in the adiabatic regime becomes

$$\omega_r=\frac{k_y\nabla_x^3\overline{\phi}+\omega_{\star e}}{1+k_\perp^2}$$

and the growth rate in the adiabatic regime becomes

$$\gamma=\frac{\omega_r^2}{\hat{\alpha}}k_\perp^2-\frac{\omega_r}{\hat{\alpha}}k_y\nabla_x^3\overline{\phi}.$$

Clearly, the vorticity gradient induces a frequency shift and reduces the growth rate. Now $R_n^{(i)}$ to leading order in $\frac{1}{\hat{\alpha}}$ in the adiabatic regime becomes

$$R^{(i)}_{nk}=-\frac{k_\perp^2\omega_r}{\hat{\alpha}}=-\frac{k_\perp^2}{\hat{\alpha}}\frac{k_y\nabla_x^3\overline{\phi}+\omega_{\star e}}{1+k_\perp^2}.$$

Hence the particle flux can be expressed as

$$\begin{aligned}\Gamma_{nx}&=\sum_k k_y\frac{k_\perp^2}{\hat{\alpha}}\frac{k_y|\phi_k|^2}{1+k_\perp^2}\left[\nabla_x^3\overline{\phi}-\nabla_x\overline{n}\right]\\ &=l_1\nabla_x^3\overline{\phi}-l_2\nabla_x\overline{n}.\end{aligned}$$

Next, the vorticity flux $\mathcal{S}=\nabla_x\Pi_{xy}$ is obtained as

$$\mathcal{S}=-\chi_y\nabla_x^3\overline{\phi}+\mathcal{S}^{res}$$

where the diffusivity is $\chi_y = \sum_k \frac{k_y^2\gamma|\phi_k|^2}{|\omega|^2}$ and the residual vorticity flux is

$$\mathcal{S}^{res} = \Re\sum_k \frac{k_y\hat{\alpha}}{\omega}\frac{\omega_{\star e}-\omega}{\omega+i\hat{\alpha}}|\phi_k|^2.$$

The residual flux must be expandable in the form

$$\mathcal{S}^{res} = L_1\nabla_x^3\overline{\phi} - L_2\nabla_x\overline{n}$$

where the first term represents negative diffusion of zonal vorticity and hence, accounts for the modulational growth of zonal flow. The second term is the density gradient dependent residual flux, which vanish in the limit $\alpha\to\infty$. So now the cross-correlation $\langle\Gamma_{nx}\nabla_x^3\bar{\phi}\rangle$ becomes

$$\langle\Gamma_{nx}\nabla_x^3\bar{\phi}\rangle = l_1\sum_q q_x^6\left\langle|\overline{\phi}_q|^2\right\rangle + l_2\Re\sum_q q_x^4\langle\overline{n}_q\overline{\phi}_q\star\rangle$$

and the cross-correlation $\langle\nabla_x\Pi_{xy}\nabla_x\bar{n}\rangle$ becomes

$$\langle\mathcal{S}\nabla_x\bar{n}\rangle = -(L_1-\chi_y)\Re\sum_q q_x^4\langle\overline{n}_q\overline{\phi}_q\star\rangle + L_2\sum_q q_x^2\left\langle|\overline{n}_q|^2\right\rangle.$$

In the adiabatic regime [64]

$$\chi_y = \sum_k\frac{k_y^2}{\hat{\alpha}}\frac{k_\perp^2}{1+k_\perp^2}|\phi_k|^2$$

$$\mathcal{S}^{res} = -\sum_k\frac{k_y^2}{\hat{\alpha}}\frac{k_\perp^2}{1+k_\perp^2}|\phi_k|^2\nabla_x\overline{n}.$$

Hence the evolution equation for zonal cross-correlation obtained from quasilinear calculations becomes

$$\frac{\partial}{\partial t}\Re\langle\bar{n}_q\bar{\phi}_q^\star\rangle - (\mu+D_n)q_x^2\Re\langle\bar{n}_q\bar{\phi}_q^\star\rangle + \left(l_2q_x^4-(L_1-\chi_y)q_x^4\right)\Re\langle\overline{n}_q\overline{\phi}_q\star\rangle + l_1q_x^6\left\langle|\overline{\phi}_q|^2\right\rangle + L_2q_x^2\left\langle|\overline{n}_q|^2\right\rangle = 0. \quad (84)$$

This is morphologically the same as equation (77), obtained by spectral calculations. Similarly an equation for $\Im\langle\bar{n}_q\bar{\phi}_q^\star\rangle$ can be obtained from the evolution equation for correlation $\langle\nabla_x\overline{n}\nabla_x^2\overline{\phi}\rangle$, which is similar to (82).

This sub-section elucidated the zonal cross-correlation, it is physics content and what spectral transfer process determines it. The zonal cross-correlation is of potential significance to layering or staircase structure, as it sets the relative phasing of shear layers and regions of ∇n steepening. Further analysis of zonal cross-correlation in layering in model boundary value problems is clearly necessary. The results of spectral calculations for zonal flows and corrugations excitation and their interaction are concisely summarized in table 1, below.

4. Wave kinetic analysis

Here we present adiabatic theory of zonal modes generation and compare the results with spectral calculations presented in the previous section.

4.1. Zonal mode equations

The zonal mode equations are obtained by flux surface average of the turbulence equations.

$$\frac{\partial}{\partial t}\nabla_x^2\overline{\phi} - \mu\nabla_x^4\overline{\phi} = \nabla_x^2\Re\int d\vec{k}k_yk_x|\phi_k|^2 \quad (85)$$

and

$$\frac{\partial}{\partial t}\overline{n} - D_n\nabla^2\overline{n} = -\nabla_x\Re\int d\vec{k}ik_yR_n|\phi_k|^2. \quad (86)$$

In a system turbulence and zonal modes co-exist, the modulations of micro-scale fields by mesoscale zonal modes are adiabatic, and so conserve wave action density $N_k = E_k/\omega_{rk}$, where E_k is the energy density of the kth wavey mode, with real frequency ω_{rk}. The action density $N(\vec{k},\vec{x},t)$ may be thought of as a wave population density—analogous to phase space density. The action density of the wavy mode has the form of $N\left(|\phi_k|^2,|n_k|^2\right)$ which using the linear Fourier amplitude relations can be expressed as $N\left(|\phi_k|^2\right)$. Then the modulated fluxes can be expressed in terms of modulation of action density via $\delta|\phi_k|^2 = C_k\delta N_k$. For the drift wave turbulence described by Hasegawa–Wakatani equations $C_k = \omega_{rk}/\left(k_\perp^2+|R_{nk}|^2\right)$, where R_{nk} is density—potential response function $R_{nk} = \frac{\omega_{\star e}+i\hat{\alpha}_k}{\omega_k+i\hat{\alpha}_k}$. The wave kinetic equation [13, 65] describing the evolution of action density is given by

$$\frac{\partial N_k}{\partial t} + \frac{\partial\omega_{rk}}{\partial\vec{k}}\cdot\frac{\partial N_k}{\partial\vec{X}} - \frac{\partial\omega_{rk}}{\partial\vec{X}}\cdot\frac{\partial N_k}{\partial\vec{k}} = \gamma_kN_k - \Delta\omega N_k^2 \quad (87)$$

where ω_{rk} and γ_k are the real frequency and growth rate in the presence of slowly varying mesoscale zonal modes. The first term on the right hand side is the linear growth and the second term is the eddy damping due to non-linear interaction which are local in k. Growth and non-linear damping balance to yield the steady state population via $\gamma_kN_k - \Delta\omega N_k^2 = 0$, in the absence of modulations. For stability analysis we make a Chapman–Enskog expansion of N_k; $N_k = \langle N_k\rangle + \delta N_k$, where $\langle N_k\rangle$ is the slowly varying mean wave action density and δN_k is the perturbation induced by the gradients of $\langle N_k\rangle$ in the phase space $\left(\vec{X},\vec{k}\right)$. The linearized wave kinetic equation for δN_k becomes

$$\left(\frac{\partial}{\partial t} + \vec{v}_{gk}\cdot\frac{\partial}{\partial\vec{X}} + \gamma_k\right)\delta N_k = \frac{\partial\delta\omega_{rk}}{\partial\vec{X}}\cdot\frac{\partial\langle N_k\rangle}{\partial\vec{k}} + \delta\gamma_k\langle N_k\rangle. \quad (88)$$

Assuming $\psi = \psi_qe^{(-i\Omega t+q_xX)}$ where $\psi_q = \{\delta N_{k,q},\phi_q,n_q\}$ the wave kinetic equation yields

$$\delta N_{k,q} = \mathcal{R}_{k,q}\left(\frac{\partial\delta\omega_{rk}}{\partial X}\frac{\partial\langle N_k\rangle}{\partial k_x} + \delta\gamma_k\langle N_k\rangle\right) \quad (89)$$

Table 1. Summary of zonal flow and corrugations interactions.

Process	Impact	Key Physics	Result
(A) Zonal flow: Vorticity equation—Polarization charge flux			
Polarization beat noise	Seeds zonal flow	Polarization flux correlation, positive definite	Equation (64)
Zonal flow response (comparable to noise)	Drives zonal shear using DW energy	Non-local inverse transfer in k_x, Negative viscosity	Equation (68)
Zonal shear straining of small scale	Regulates waves via straining	Stochastic refraction straining waves, induced diffusion to high k_x	Equation (45)
(B) Density corrugations: Density equation—Particle flux			
Process	Impact	Key Physics	Result
Density advection beat noise	Seeds density corrugation	Advection beats due to non-adiabatic density	Equation (72)
Zonal corrugations response	Dampes and regulates density corrugations	Non-local forward transfer in k_x, Positive diffusivity, turbulent mixing weak for $\alpha \gg 1$	Equation (71)
Zonal shear straining of small scale	Regulates waves via straining	Stochastic refraction straining waves, induced diffusion to high k_x	Equation (47)
(C) Zonal cross-correlation: Vorticity and density transport processes			
Process	Impact	Key Physics	Result
Noise and response	Sets corrugation—shear layer correlation; staircase states	Real cross-correlation spectrum +ve when growth of zonal intensity exceeds damping rate of corrugation, otherwise negative	Equation (78)

where the propagator $\mathcal{R}_{k,q}$ is given by

$$\mathcal{R}_{k,q} = \frac{i}{\Omega_q - q_x v_{gx} + i|\gamma_k|}. \tag{90}$$

Now the frequency modulation by the zonal modes can be obtained as

$$\delta\omega_{rk} = k_y \nabla_X \overline{\phi} - k_y \frac{\partial \omega_{rk}}{\partial \omega_{\star e}} \nabla_X \overline{n}. \tag{91}$$

Similarly the growth rate modulation can be obtained as

$$\delta\gamma_k = -k_y \frac{\partial \gamma_k}{\partial \omega_{\star e}} \nabla_X \overline{n} \tag{92}$$

where

$$\frac{\partial \omega_{rk}}{\partial \omega_{\star e}} = \frac{\hat{\alpha}\left(2k_\perp^2 \gamma + \hat{\alpha}\left(1+k_\perp^2\right)\right)}{\left|2k_\perp^2 \omega + i\hat{\alpha}\left(1+k_\perp^2\right)\right|^2}$$

and

$$\frac{\partial \gamma_k}{\partial \omega_{\star e}} = \frac{\hat{\alpha} 2k_\perp^2 \omega_r}{\left|2k_\perp^2 \omega + i\hat{\alpha}\left(1+k_\perp^2\right)\right|^2}.$$

Finally, using the above expressions for action density modulation, frequency and growth rate modulations, the equation for zonal vorticity becomes

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_x^2 \overline{\phi} - \mu \nabla_x^4 \overline{\phi} &= \nabla_x^2 \int d\vec{k} k_y k_x C_k \mathcal{R}_{k,q} \\ &\times \left[k_y \left\{\nabla_X^2 \overline{\phi} - \frac{\partial \omega_{rk}}{\partial \omega_{\star e}} \nabla_X^2 \overline{n}\right\} \frac{\partial \langle N_k \rangle}{\partial k_x} - k_y \frac{\partial \gamma_k}{\partial \omega_{\star e}} \nabla_X \overline{n} \langle N_k \rangle\right] \\ &= \nabla_x^2 \int d\vec{k} k_y k_x C_k \left[\mathcal{R}_{k,q}^{(r)} k_y \left\{\nabla_X^2 \overline{\phi} - \frac{\partial \omega_{rk}}{\partial \omega_{\star e}} \nabla_X^2 \overline{n}\right\} \right. \\ &\left. \frac{\partial \langle N_k \rangle}{\partial k_x} - i\mathcal{R}_{k,q}^{(i)} k_y \frac{\partial \gamma_k}{\partial \omega_{\star e}} \nabla_X \overline{n} \langle N_k \rangle\right].\end{aligned} \tag{93}$$

Similarly the equation for zonal density becomes

$$\begin{aligned}\frac{\partial}{\partial t}\overline{n} - D_n \nabla_x^2 \overline{n} &= \nabla_x \int d\vec{k} k_y R_{nk}^{(i)} C_k \mathcal{R}_{k,q} \\ &\times \left[k_y \left\{\nabla_X^2 \overline{\phi} - \frac{\partial \omega_{rk}}{\partial \omega_{\star e}} \nabla_X^2 \overline{n}\right\} \frac{\partial \langle N_k \rangle}{\partial k_x} - k_y \frac{\partial \gamma_k}{\partial \omega_{\star e}} \nabla_X \overline{n} \langle N_k \rangle\right] \\ &- \nabla_x \int d\vec{k} l_1 \nabla_x^3 \overline{\phi} C_k \langle N_k \rangle \\ &= -\nabla_x \int d\vec{k} k_y R_{nk}^{(i)} C_k \mathcal{R}_{k,q}^{(r)} k_y \frac{\partial \gamma_k}{\partial \omega_{\star e}} \langle N_k \rangle \nabla_X \overline{n} \\ &- \nabla_x \int d\vec{k} l_1 \nabla_x^3 \overline{\phi} C_k \langle N_k \rangle.\end{aligned} \tag{94}$$

Note that the first term on the right hand side in the above equation results from the flux modulation due to modulation of action density, and the second term results from flux modulation via modulation of the wavy density-potential response function $R_n^{(i)}$. The terms proportional to $\frac{\partial \langle N_k \rangle}{\partial k_x}$ vanished because the integrand is odd in k_y. That is, action density modulation due to frequency modulation does not contribute to flux modulation. Hence particle flux modulation is independent of the spectral gradient. Note that particle flux modulation occurs via growth rate modulation of action density and frequency modulation of $R_n^{(i)}$ whereas the Reynolds stress modulation occurs via frequency and growth rate modulation of action density, only. The particle flux modulation is independent of the spectral slope, whereas the Reynolds stress modulation depends upon spectral slope. The above equation shows that zonal density modulations are damped for $\alpha > 1$. The above zonal mode equations show that the evolutions of zonal flow and corrugations are coupled, but different, and are consistent with the results obtained by the spectral calculations.

4.2. Back reaction of zonal modes on drift wave turbulence

While the zonal flows and corrugations are generated by turbulence, they also react back on turbulence via random refraction (shearing and corrugation) in k-space. The back reaction of the zonal modes on the turbulence can be studied by taking the average of the WKE (87).

$$\frac{\partial\langle N_k\rangle}{\partial t} = \left\langle \frac{\partial\omega_{rk}}{\partial\vec{X}}\cdot\frac{\partial N_k}{\partial\vec{k}}\right\rangle + \langle\gamma_k N_k\rangle - \Delta\omega\left\langle N_k^2\right\rangle$$
$$= \left\langle \frac{\partial\delta\omega_{rk}}{\partial\vec{X}}\cdot\frac{\partial\delta N_k}{\partial\vec{k}}\right\rangle + \langle\delta\gamma_k\delta N_k\rangle - \Delta\omega\langle\delta N_k\delta N_k\rangle. \quad (95)$$

Using the linear response of δN_k for the zonal mode perturbation, a quasilinear expression for the first term on the right hand side of the above equation is obtained

$$\left\langle \frac{\partial\delta\omega_{rk}}{\partial\vec{X}}\cdot\frac{\partial\delta N_k}{\partial\vec{k}}\right\rangle = \frac{\partial}{\partial k_x}\left[\left\langle\frac{\partial\delta\omega_{rk}}{\partial X}\mathcal{R}_{kq}\frac{\partial\delta\omega_{rk}}{\partial X}\right\rangle\frac{\partial\langle N_k\rangle}{\partial k_x} + \left\langle\frac{\partial\delta\omega_{rk}}{\partial X}\mathcal{R}_{kq}\delta\gamma_k\right\rangle\langle N_k\rangle\right].$$

Similarly,

$$\langle\delta\gamma_k\delta N_k\rangle = \left\langle\frac{\partial\delta\omega_{rk}}{\partial X}\mathcal{R}_{kq}\delta\gamma_k\right\rangle\frac{\partial\langle N_k\rangle}{\partial k_x} + \langle\delta\gamma_k\mathcal{R}_{kq}\delta\gamma_k\rangle\langle N_k\rangle.$$

Eventually, we arrive at the following evolution equation for the mean action density under the influence of zonal modes.

$$\frac{\partial\langle N_k\rangle}{\partial t} = \frac{\partial}{\partial k_x}\left[D_{kk}\frac{\partial\langle N_k\rangle}{\partial k_x} + V_k\langle N_k\rangle\right] + V_k\frac{\partial\langle N_k\rangle}{\partial k_x} + \Gamma_k\langle N_k\rangle. \quad (96)$$

Equation (96) is a convection–diffusion equation, with k_x-space diffusivity D_{kk} given by

$$D_{kk} \equiv \left\langle\frac{\partial\delta\omega_{rk}}{\partial X}\mathcal{R}_{kq}\frac{\partial\delta\omega_{rk}}{\partial X}\right\rangle = \int d\vec{q}q^2\mathcal{R}_{k,q}^{(r)}|\delta\omega_{kq}|^2$$
$$= \int d\vec{q}\mathcal{R}_{k,q}^{(r)}k_y^2q_x^4\left|\phi_q - \frac{\partial\omega_{rk}}{\partial\omega_{\star e}}n_q\right|^2, \quad (97)$$

the convection speed V_k is

$$V_k \equiv \left\langle\frac{\partial\delta\omega_{rk}}{\partial X}\mathcal{R}_{kq}\delta\gamma_k\right\rangle = -\Re\int d\vec{q}i\mathcal{R}_{k,q}k_yq^3$$
$$\times\left(\phi_q - \frac{\partial\omega_{rk}}{\partial\omega_{\star e}}n_q\right)k_y\frac{\partial\gamma_k}{\partial\omega_{\star e}}n_{-q}$$
$$= \int d\vec{q}\frac{v_{gx}k_y^2q_x^4}{|\Omega_q - q_xv_{gx} + i\gamma_k|^2}\left(\phi_q - \frac{\partial\omega_{rk}}{\partial\omega_{\star e}}n_q\right)\frac{\partial\gamma_k}{\partial\omega_{\star e}}n_{-q} \quad (98)$$

and the non-linear growth rate Γ_k becomes

$$\Gamma_k \equiv \langle\delta\gamma_k\mathcal{R}_{kq}\delta\gamma_k\rangle = \int d\vec{q}\mathcal{R}_{k,q}^{(r)}|\delta\gamma_{kq}|^2$$
$$= \int d\vec{q}\mathcal{R}_{k,q}^{(r)}q_x^2k_y^2\left(\frac{\partial\gamma_k}{\partial\omega_{\star e}}\right)^2|n_q|^2. \quad (99)$$

Equation (96) describe how a zonal flow shear and density corrugations lead to diffusion of turbulence in k-space. While zonal flow shear only diffuses turbulence in k-space, density corrugations play a role in both in k-space diffusion and non-linear growth of turbulence. The expression for diffusivity D_{kk} reveal that density corrugation can enhance or reduce turbulence diffusion depending on the phase relation between zonal potential and zonal density—i.e. zonal cross correlation!. It is also interesting to note that density corrugation contributes to convection of turbulence in k-space. Clearly, the sign of the convection speed V_k depends on the zonal cross-correlation. The non-linear growth rate Γ_k due to linear growth modulation by density corrugation injects energy back into the turbulence, locally. Thus, there is a competition between the random shearing of the zonal flow as a saturation mechanism, and energy reintroduction into the turbulence via density corrugations.

Comparison with the spectral calculations shows that turbulence kinetic energy and internal energy diffusivity scale as the square of zonal shear $q^4\phi_q^2$, whereas WKE analysis shows that the action density diffusivity scales as $q_x^4\left|\phi_q - \frac{\partial\omega_{rk}}{\partial\omega_{\star e}}n_q\right|^2$. This clearly shows the important role of zonal cross-correlation in setting the k-space diffusion of action density. In contrast, the role of zonal cross-correlation in spectral energy diffusivity is not immediately clear from the spectral analysis so far. Further investigation of this point is necessary.

To put sections 3 and 4 in perspective, table 2 below gives a comparison of wave kinetic theory results for zonal modes (flows and corrugations) with the spectral equation calculations presented earlier in this paper.

5. Predator prey dynamics with non-linear zonal noise

Here we study the effect of zonal noise on the predator prey dynamics of turbulence energy and zonal flow energy. Here, we follow the model of [1] which evolves turbulence energy and zonal flow energy in 0D for the strongly adiabatic limit. The turbulence energy ε evolves as

$$\frac{\partial\varepsilon}{\partial t} = \gamma\varepsilon - \sigma E_v\varepsilon - \eta\varepsilon^2 \quad (100)$$

Table 2. Comparison of spectral and wave kinetic results.

Physical effect	Spectral theory (1)	Adiabatic theory (2)	Comments
Zonal flow modulation	Viscosity $\sum_{\vec{q}} q_y^2 \Theta^{(r)}_{k,-q,q} q_x \times \frac{\partial}{\partial q_x}\left[\left(a_{-q}q^2 + b_{-q}R^{\star}_{nq}\right)^{(r)} I_q\right]$	Viscosity $\int d\vec{k} k_y k_x C_k \mathcal{R}^{(r)}_{k,q} k_y \frac{\partial\langle N_k\rangle}{\partial k_x}$ $-$ve for $\frac{\partial\langle N_k\rangle}{\partial k_x} < 0$	For $\alpha\to\infty$, both predict same growth rate due to $-$ve viscosity with $\Theta_{k,-q,q}\leftrightarrow\mathcal{R}_{k,q}$.
Corrugations modulation	Particle diffusivity $\sum_{\vec{q}} \Theta^{(r)}_{k,-q,q} q_y^2 \frac{4q^4}{(1+q^2)^2}\frac{1}{\alpha_q^2} I_q$ +ve for $\alpha > 1$, scales as $\frac{1}{\alpha^2}$.	Particle diffusivity $-\int d\vec{k} k_y R^{(i)}_{nk} C_k \mathcal{R}^{(r)}_{k,q} k_y \frac{\partial\gamma_k}{\partial\omega_{\star e}}\langle N_k\rangle$ +ve for $\alpha > 1$, scales as $\frac{1}{\alpha^2}$.	Both predict same damping rate for corrugation due to +ve diffusivity with $\Theta_{k,-q,q}\leftrightarrow\mathcal{R}_{k,q}$.
Induced diffusion	Kinetic energy diffusivity $\sum_q 4k_y^2\left(\frac{k}{k_\perp}\right)^4 q_x^2 E_q \Theta^{(r)}_{kkq} a^r_k$, Internal energy diffusivity $\sum_q k_y^2 q_x^4 \left\langle\lvert\phi_q\rvert^2\right\rangle \Theta^{(r)}_{kkq} d^r_k$	Mean action density diffusivity $\int d\vec{q} \mathcal{R}^{(r)}_{k,q} k_y^2 q_x^4 \left\lvert\phi_q - \frac{\partial\omega_{rk}}{\partial\omega_{\star e}} n_q\right\rvert^2$ Density corrugations decrease (increase) action diffusivity when the zonal density and potential are correlated (anti-correlated).	Both kinetic and internal energy diffuse by zonal shear energy. Role of zonal cross-correlation unclear in (1), ignored for simplicity.

where the first term on the right hand side represents linear growth of turbulence with growth rate γ. The second term represents turbulence damping due to diffusion induced by zonal flow in the k_x-space. The third term represents the nonlinear damping of turbulence, by self-interaction. The zonal flow energy E_v evolves as

$$\frac{\partial E_v}{\partial t} = \sigma\varepsilon E_v - \gamma_d E_v + \beta\varepsilon^2 \tag{101}$$

where the first term on the right hand side represents modulational growth of zonal flow and the second term represents collisional damping of zonal flow, with damping rate γ_d. The third term $\beta\varepsilon^2$ on the right hand side of equation (101) represents drive by the zonal noise, a new element in the model as presented here. It is this term which makes our predator–prey model different from previous incarnations of the model [1]. The parameters of this model are $\gamma = \frac{k_\perp^2}{\hat{\alpha}}\frac{\omega_k^{r2}}{1+k_\perp^2}$, $\sigma = \sum_q 2k_x^2\Theta^{(r)}_{k,-q,q}$, $\beta = \sum_q 4k_x^2 q_y^{-2} q_x^2 \Theta^{(r)}_{k,-q,q}$, $\gamma_d = \mu k_x^2$.

5.1. Fixed point analysis

The above equations yield, for steady state:

$$\sigma E_v = \gamma - \eta\varepsilon \tag{102}$$

and

$$(\sigma\varepsilon - \gamma_d) E_v + \beta\varepsilon^2 = 0. \tag{103}$$

Defining $\varepsilon_1 = \gamma_d/\sigma$ and $\varepsilon_2 = \gamma/\eta$ and using the above equations, the fixed points are obtained by the roots of the following equation

$$\left(1 - \frac{\beta}{\eta}\right)\varepsilon^2 - \varepsilon(\varepsilon_1 + \varepsilon_2) + \varepsilon_1\varepsilon_2 = 0,$$

which are:

$$\varepsilon^{\pm} = \frac{(\varepsilon_1+\varepsilon_2) \pm \sqrt{(\varepsilon_1+\varepsilon_2)^2 - 4\left(1-\frac{\beta}{\eta}\right)\varepsilon_1\varepsilon_2}}{2\left(1-\frac{\beta}{\eta}\right)}. \tag{104}$$

The corresponding zonal flow energies are

$$E_v^{\pm} = \sigma^{-1}\left(\gamma - \eta\varepsilon^{\pm}\right). \tag{105}$$

Note that for the case without noise $\beta = 0$, $\varepsilon_0^+ = \varepsilon_2$, $\varepsilon_0^- = \varepsilon_1$, $E_{v0}^+ = 0$ and $E_{v0}^- = \sigma^{-1}\eta(\varepsilon_2 - \varepsilon_1)$. It is straight forward to check that the fixed point $\left(\varepsilon_0^-, E_{v0}^-\right) = \left(\varepsilon_1, \sigma^{-1}\eta(\varepsilon_2-\varepsilon_1)\right)$ is stable. Clearly, there is a threshold in growth rate γ for excitation of zonal flow in the noise free case. This threshold is

$$\gamma > \eta\frac{\gamma_d}{\sigma} \tag{106}$$

and can be linked to a threshold in edge gradients or flux (power). Is there a threshold in γ for zonal flow excitation with noise? Using equations (104) and (105), we see:

$$\gamma > \eta\frac{(\varepsilon_1+\varepsilon_2) \pm \sqrt{(\varepsilon_1+\varepsilon_2)^2 - 4\left(1-\frac{\beta}{\eta}\right)\varepsilon_1\varepsilon_2}}{2\left(1-\frac{\beta}{\eta}\right)}$$

which implies

$$\frac{\gamma^2}{\eta^2}\left[\left(1 - 2\frac{\beta}{\eta}\right)^2 - 1\right] > 0.$$

This clearly shows that, with noise, there is no threshold in γ for zonal flow excitation. This is consistent with numerical solutions plotted in figure 1. The phase plane in figure 1 is obtained by performing a linear growth rate scan with noise strength as a parameter. A linear growth rate scan is a proxy for a power scan, as power changes the pressure gradient and hence the growth rate. The figure shows that, without noise, there is a threshold in growth rate (or power) for appearance of stable zonal flows. Below the threshold, there is only turbulence, and no zonal flows. Beyond the threshold growth rate—both turbulence and zonal flows co-exist. On ramping up the growth rate (or power), the turbulence energy increases as γ/η below the threshold, until it locks at γ_d/σ, at the threshold. Beyond the threshold, turbulence energy remains locked at the value γ_d/σ while the zonal flow energy continues to grow

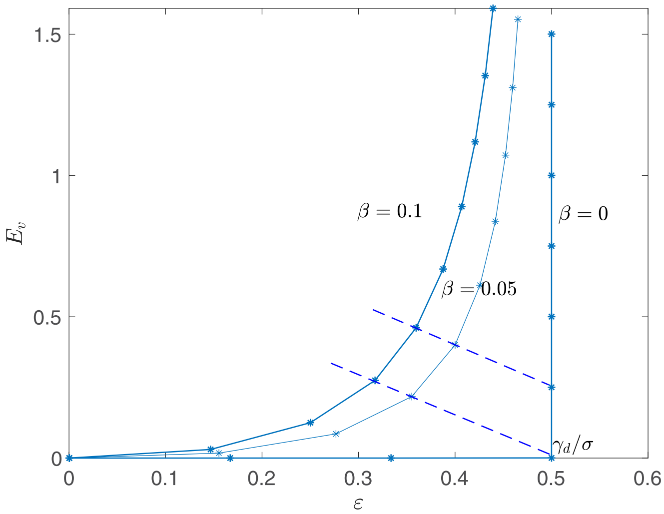

Figure 1. Zonal flow energy E_v vs turbulence energy ε in a linear growth rate γ scan with noise strength β as a parameter.

as $\sigma^{-1}\eta(\gamma/\eta - \gamma_d/\sigma)$. With noise, both zonal flow and turbulence co-exist at any value of growth rate—i,e. there is no threshold for zonal flow excitation. Both zonal flow and turbulence increase with growth rate. In this case, zonal flow energy is related to turbulence energy as $E_v = \beta\varepsilon^2/(\gamma_d - \sigma\varepsilon)$. Note that, with noise, the turbulence energy never hits the modulational instability threshold, absent noise! Significant zonal flows are generated well below the modulational instability threshold.

The next question is how does the noise free base state change with a weak noise? Taylor expanding about $\frac{\beta}{\eta} = 0$, it is straight forward to show

$$\varepsilon^{\pm} = \varepsilon_0^{\pm} \pm \frac{\beta}{\eta} \frac{\varepsilon_0^{\pm 2}}{\varepsilon_0^{+} - \varepsilon_0^{-}} + \mathcal{O}\left(\frac{\beta^2}{\eta^2}\right) \tag{107}$$

and

$$E_v^{\pm} = E_{v0}^{\pm} \mp \frac{\beta}{\eta}\frac{\eta}{\sigma} \frac{\varepsilon_0^{\pm 2}}{\varepsilon_0^{+} - \varepsilon_0^{-}} + \mathcal{O}\left(\frac{\beta^2}{\eta^2}\right). \tag{108}$$

Since $\varepsilon_0^{+} > \varepsilon_0^{-}$, the above equations show that the turbulence energy decreases and zonal energy increases with noise corresponding to the stable fixed point. This is consistent with the numerical solutions shown in figure 2. Why?—Noise feeds energy into zonal flows!

5.2. Stability of fixed points

The Jacobian of the system of equations (100) and (101) is

$$D(\varepsilon, E_v) = \begin{bmatrix} -\eta\varepsilon & -\sigma\varepsilon \\ \gamma + (-\eta + 2\beta)\varepsilon & \sigma\varepsilon - \gamma_d \end{bmatrix}. \tag{109}$$

The stability of the fixed points are determined by the eigenvalues λ of the $D(\varepsilon, E_v)$

$$\lambda^2 + \lambda\left[(\eta - \sigma)\varepsilon + \gamma_d\right] + \gamma_d\gamma + \eta\sigma\left(\frac{\beta}{\eta} - 1\right)\varepsilon^2 = 0. \tag{110}$$

The roots are obtained as

$$\lambda = \frac{-\left[(\eta - \sigma)\varepsilon + \gamma_d\right] \pm \sqrt{\left[(\eta - \sigma)\varepsilon + \gamma_d\right]^2 - 4\left[\gamma_d\gamma + \eta\sigma\left(\frac{\beta}{\eta} - 1\right)\varepsilon^2\right]}}{2}. \tag{111}$$

At $\beta = 0$ and $\varepsilon = \gamma_d/\sigma$

$$\lambda_0 = \frac{-\eta\frac{\gamma_d}{\sigma} \pm \sqrt{\left[\eta\frac{\gamma_d}{\sigma}\right]^2 - 4\eta\gamma_d\left[\frac{\gamma}{\eta} - \frac{\gamma_d}{\sigma}\right]}}{2}.$$

This shows that the steady state $\left(\varepsilon_0^{-}, E_{v0}^{-}\right)$ is stable without noise. To study effect of noise on the stability of the steady states, numerical solution of equation (102) is more convenient. The results are plotted in figure 2, which show that the fixed point stability degrades with increasing noise strength.

Hence, we see that polarization beat noise affects the predator-prey dynamics significantly, by eliminating the threshold in the linear turbulence growth rate. Zonal flows and turbulence always co-exist at any growth rate. The zonal flow energy-to-turbulence energy branching ratio increases with noise strength, as the polarization beat noise pumps energy into zonal modes.

6. Noise effect on the L–H transition

The previous section showed that polarization beat noise has a significant effect on predator-prey dynamics of zonal flows and turbulence. It eliminates the threshold (in the linear growth of turbulence) for the onset of zonal flows. This indicates that zonal noise may have an observable effect on the dynamics of the L–H transition. So here, in this section, we study the effect of noise on the dynamics of the L–H transition. We examine a $0D$ model evolving turbulence energy ε, zonal flow energy, and mean pressure gradient P for this purpose. This minimal model is an extension of the $KD03$ model ala Kim and Diamond [4]. The model is exceedingly simple. The point here is to illustrate noise effects in a familiar setting. The normalized turbulence kinetic energy $\varepsilon = q_y^2\rho_s^2 I_q / q_y^2\rho_s^2\rho^{\star 2}$ evolves as:

$$\frac{\partial\varepsilon}{\partial t} = \frac{a_1\mathcal{P}\varepsilon}{1 + a_3\mathcal{V}^2} - a_2\varepsilon^2 - \frac{a_4 v_z^2\varepsilon}{1 + b_2\mathcal{V}^2}. \tag{112}$$

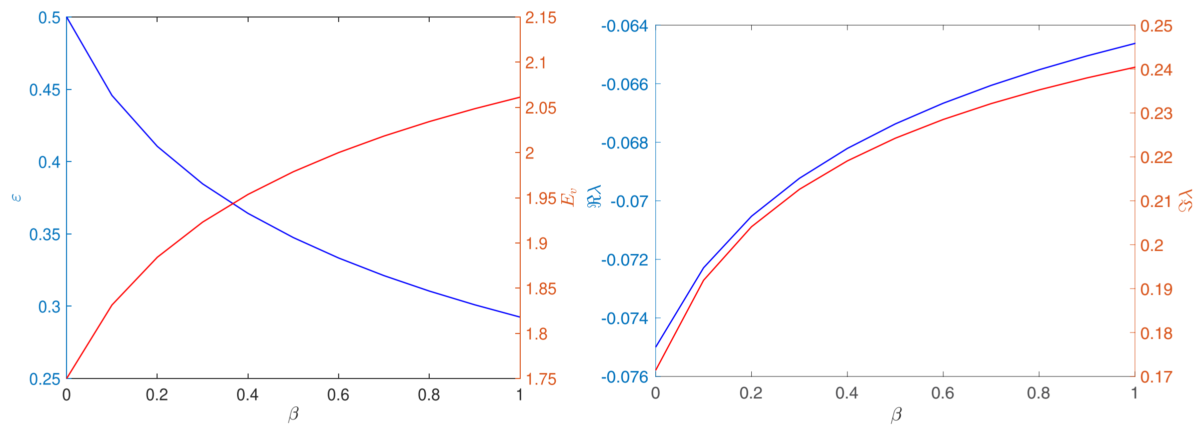

Figure 2. Left: Stable fixed points with noise strength β. Zonal flow energy increases and turbulence energy decreases with β. Right: eigenvalues λ of the stable fixed points.

Here t is time normalized by gyro-Bohm diffusion time i.e. $t \equiv tD_{GB}/a^2$, where $D_{GB} = c_i\rho_i\rho^\star$ is gyro-Bohm diffusivity and a is the minor radius. The first term on the right hand side represents linear growth of turbulence driven by pressure gradient $\mathcal{P} = a|\nabla P|/P_o$, via instability. The growth rate coefficient is normalized as $a_1 \equiv a_1 a/c_i\rho^{\star 2}$ and the non-linear damping rate coefficient is normalized as $a_2 \equiv a_2 a/c_i\rho^{\star 2}$. The factor $\frac{1}{1+a_2\mathcal{V}^2}$ represents linear growth rate reduction by mean flow shear $\mathcal{V}$. The second term represents non-linear damping of turbulence and the third term represents local damping of turbulence due to k_x-space diffusion induced by mean square zonal flow shear. The evolution of normalized zonal flow kinetic energy $v_z^2 = k_x^2\rho_s^2 I_k/k_x^2\rho_s^2\rho^{\star 2}$ is governed by

$$\frac{\partial v_z^2}{\partial t} = \frac{b_1\varepsilon v_z^2}{1+b_2\mathcal{V}^2} - b_3 v_z^2 + b_4\varepsilon^2. \qquad (113)$$

The first term on the right hand side represents modulational growth of zonal flow by Reynolds stress, where $b_1 = 2(k_x^2\rho_s^2/\rho^{\star 2})\sum_q q_y^2\rho_s^2\Theta c_s/a$ and Θ is triad interaction time, in dimensional form. The factor $\frac{1}{1+b_2\mathcal{V}^2}$ represents inhibition of modulational growth by mean flow shear. This inhibition is due to the weakening of the response of drift wave spectrum to a seed zonal flow, via the enhanced decorrelation of drift wave propagation by a mean shear flow. Note that the same suppression factor is appears in the damping, due to diffusion induced by zonal flow shear. This guarantees conservation of total energy of turbulence and zonal flow. The second term is the linear damping of zonal flow due to collisional drag. The third term, proportional to turbulence energy squared, represents the zonal noise with $b_4 = (4/\rho^{\star 2})\sum_q q_x^2\rho_s^2 q_y^2\rho_s^2\Theta(c_s/a)$. This is the unique feature of this incarnation of the KD03 model. The pressure gradient $\mathcal{P}$ evolves according to:

$$\frac{\partial \mathcal{P}}{\partial t} = -c_1\frac{\varepsilon\mathcal{P}}{1+c_2\mathcal{V}^2} - c_3\mathcal{P} + Q \qquad (114)$$

where the first term on the right hand side represents local damping by turbulent diffusion. The normalized turbulent damping coefficients are $c_1 = (a/L)^2(D_T/D_{GB})$ and $c_3 = (a/L)^2(D_{nc}/D_{GB})$, where D_T and D_{nc} are turbulent and neoclassical diffusivities and D_{GB} is gyro-Bohm diffusivity. The factor $\frac{1}{1+c_2\mathcal{V}^2}$ accounts for transport suppression due to transport cross-phase reduction by the mean flow shear. The second term represents neoclassical transport. The third term Q is a normalized source function gradient that represents input external power, $Q = a^2\nabla S_p/P_0^2 c_i\rho^{\star 2}$. Here S_p is the actual pressure (i.e. heat) source function. Finally, the normalized mean flow shear $\mathcal{V} \equiv V_E' a/\rho^\star c_i$ is related to the pressure gradient $\mathcal{P}$ through the diamagnetic part of radial force balance

$$\mathcal{V} = -\mathcal{P}^2 \qquad (115)$$

where coupling to mean poloidal and toroidal flows are ignored and a constant ion temperature profile is assumed for simplicity. Note that this model is an outgrowth of, and yet significantly different from, the *KD*03 model, in the sense that it not only considers the effect of zonal noise but also includes the effect of mean $E\times B$ induced suppression of turbulence growth, modulational zonal growth and transport cross-phase reduction. These physically motivated modifications allow producing an H-mode with residual turbulence, and manifest hysteresis phenomenon in a cyclic power ramp.

The input power Q is the main control parameter of this model. The noise strength b_4 is a subsidiary control parameter which facilitate study of effect of noise on L–H transition. The equations (112)–(115) are solved numerically for the $Q = 0.01t$, and assuming constant values for the parameters a_i, $\boldsymbol{b_i}$ and $\boldsymbol{c_i}$. Figure 3 shows the evolution of turbulence energy, zonal flow energy and pressure gradient as the input power is ramped up, with noise strength b_4 as a parameter. First, we discuss the noise-free case i.e. $b_4 = 0$. Clearly there are three distinct stages. The L-mode, I(intermediate)-phase and the H mode. L mode is the initial stage in which (as the input power Q ramped up from zero), the mean pressure gradient $\mathcal{P}$ steepens and excites turbulence from linear instability. Notice that there is no zonal flow in the L-mode. Upon further heating, turbulence continues to grow and excites zonal flows when the input power exceeds a threshold set by turbulence level and flow damping. When the turbulent drive becomes

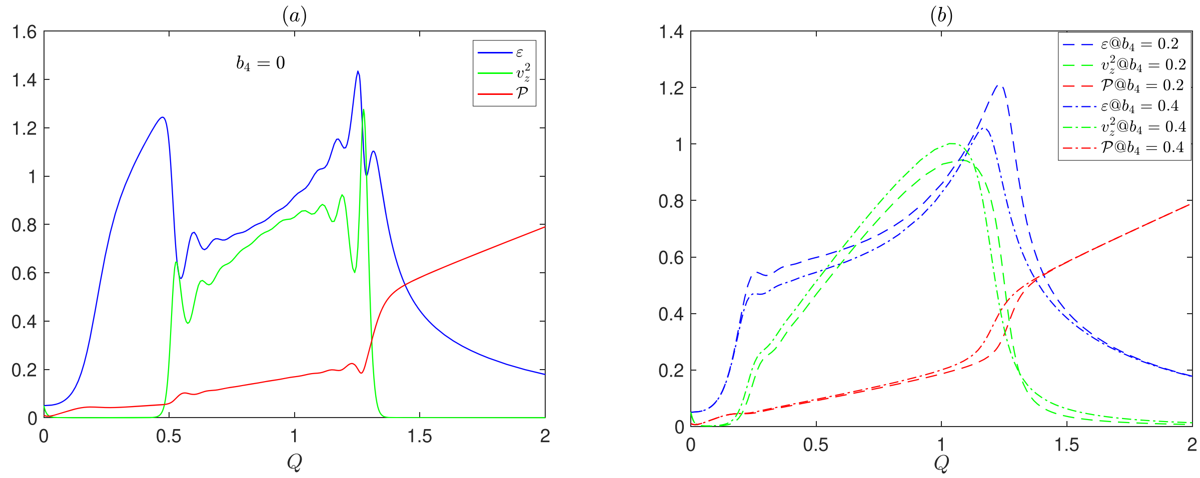

Figure 3. (a) L–H transition dynamics without noise. Notice that zonal flow exist only within the I-phase and residual turbulence exists in H-mode. (b) L–H transition dynamics with noise. Notice that zonal flow exists at any power, but is most prominent within the I-phase. Noise eliminates the threshold for zonal flow appearance, increases the duration of I-phase, reduces the turbulence energy and reduces the ultimate power threshold for L–H transition. Parameters are $a_1 = 1$, $a_2 = 0.2$, $a_3 = 0.7$, $a_4 = 0.7$, $b_1 = 1.5$, $b_2 = 0.7$, $b_3 = 1$, $c_1 = 1$, $c_2 = 0.7$ and $c_3 = 0.5$.

strong enough to overcome flow damping, it generates zonal flows by Reynolds stress. The turbulence energy overshoots dramatically before exciting the zonal flow. Turbulence and zonal flows then form a self-regulating system, as the shearing by zonal flows damps the turbulence. The first appearance of zonal flow marks the beginning of the I-phase. In the I-phase, zonal flows and turbulence compete, and oscillatory behavior emerge. A gradual increase in the turbulence energy is noticed in the I-phase. This is due to the reduction in the zonal flow growth by the mean shear flow, which strengthens the growth of turbulence. The behavior of the turbulence envelope in the I-phase is given by the stationary solution of equation (113) i.e. $\varepsilon = b_3\left(1 + b_2\mathcal{V}^2\right)/b_1$, which increases as the pressure gradient increases with Q. At sufficiently high Q, the system bifurcates into H mode, when turbulence energy drops suddenly as the pressure gradient jumps up and zonal flows disappear. This is the H mode, with non-zero residual turbulence. This is more realistic than the Quiescent H mode with no turbulence, predicted by the KD03 model. After the transition to H mode, the pressure gradient continues to rise and turbulence energy continues to fall. The pressure gradient is primarily set by the neoclassical transport, since the turbulent transport is drastically reduced due to cross-phase reduction by the strong mean shear. Further heating may excite MHD instabilities, which are not modeled here.

Next, we discuss the case with finite zonal noise ($b_4 \neq 0$). The dashed and dotted dashed curves in figure 3 correspond to finite noise. There are several important differences as compared to the case without noise.

(a) The most striking change is that significant zonal flows appear much earlier than for the modulational instability threshold without noise. In fact, now there is no clear threshold in Q (unless there is a threshold for linear instability, which is assumed not to be the case here) for zonal flow appearance.
(b) The turbulence level is reduced, there is no overshoot, and zonal flows are enhanced.
(c) The I-phase oscillations are smaller.
(d) The transition to H-mode (marked by a sharp jump in pressure gradient) occurs at lower Q—i.e. the power threshold is lower. This is because zonal noise couples more fluctuation energy to benign zonal modes.
(e) Zonal flow remains small but finite in H mode.

The zonal flow in the early phase is noise driven. The initial exponential rise of the zonal flow tracks the exponential rise of turbulence energy. This phase of exponentially rising turbulence is the L mode. On increasing power, the system enters the I-phase, marked by an approximately linear growth of turbulence and zonal flow energy. Notice that the I-phase with noise begins with small overshoot of turbulence. On further heating, there comes a point when the steepening of the pressure profile starts to accelerate, and mean shear begins to overtake zonal flow shear. The modulational growth is then reduced by the mean shear and the zonal flow beings to decay. Notice that turbulence intensity begins to roll down at a higher Q than does the zonal flow. This transient non-linear rise of turbulence is due to depletion of zonal flows by mean shear. In the H mode, the residual turbulence emits zonal noise and hence zonal flow energy tracks the turbulence energy. So zonal flows are present in all the three phases (i.e. at any Q)!

6.1. Hysteresis

Figure 4 shows the evolution of turbulence energy, zonal flow energy and pressure gradient in a cyclic power ramp at a finite noise strength. Clearly, such a cyclic evolution exhibits hysteresis. Turbulence, zonal flow and pressure gradient do not retreat along the same respective curves, and the H–I back transition occurs at a lower Q than that of I–H. Notice that the back transition is associated with an oscillating I-phase,

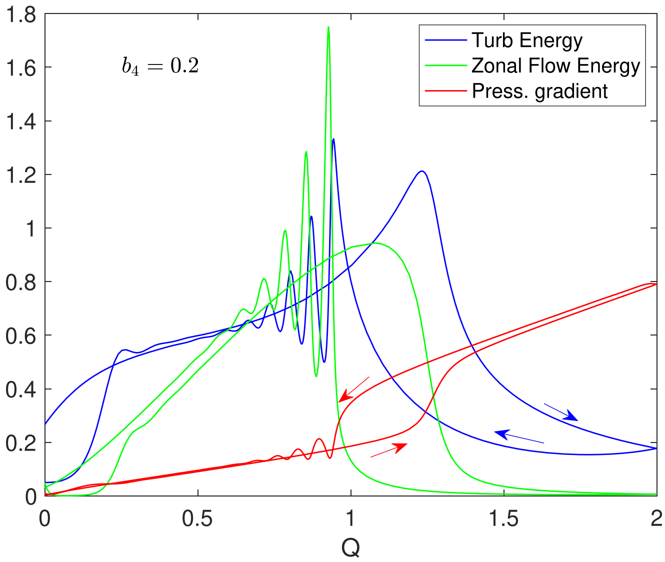

Figure 4. The H–L back transition is accompanied by hysteresis. In presence of zonal noise, hysteresis is robust with respect to variations in initial conditions and the point of retreat (in Q) in the H mode.

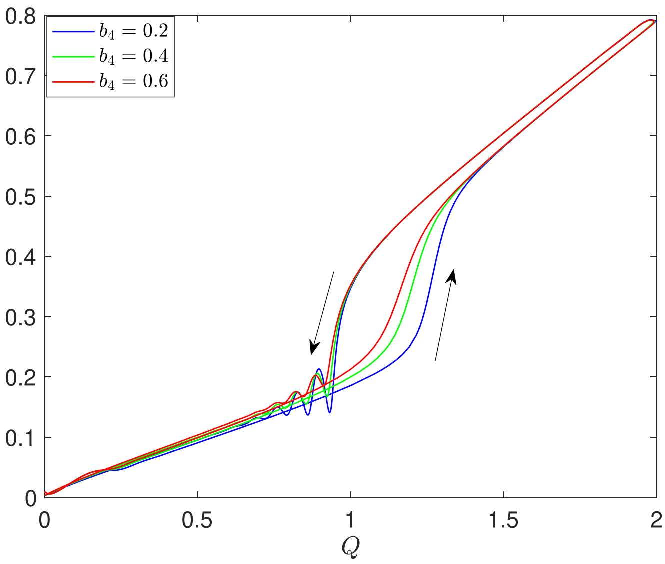

Figure 5. Noise reduces the area enclosed in the hysteresis curve.

while the I-phase in the forward transition is not oscillatory. The hysteresis, with noise, is robust with respect to variations in the initial conditions and the depth of the H mode (i.e. how far in to the H mode the power ramp down begins). Whereas without noise, the hysteresis is sensitive to the initial condition and the depth of the H mode i.e. hysteresis depends on where in H mode the power ramp-down begins. In the absence of noise, hysteresis is not robust.

Figure 5 shows effect of increasing noise strength on the hysteresis in flux and pressure gradient. It can be seen that the threshold power for both forward and backward transition decreases at different rates with noise strength. As a result the area enclosed by the hysteresis curve decreases with noise. Also the I-phase oscillations in the back transition are reduced with noise strength.

This section elucidated the effect of zonal noise on the dynamics of L–H transition and on L–H hysteresis in a cyclic power ramp. The most significant effect is that noise-driven zonal flows appear in all modes of the discharge. Zonal flows appear far below the modulational stability threshold. Zonal noise increases the extent of the I-phase, reduces overall turbulence energy and reduces the threshold power for the I–H transition. Zonal noise makes hysteresis phenomenon robust and reduces the area enclosed by the hysteresis curve in a cyclic power ramp.

7. Conclusions and discussion

In this paper, we presented a unified theory of zonal mode dynamics. We linked the mechanism of zonal flow excitation by polarization flux beat noise to that of modulational instability (i.e. instability due to negative turbulent viscosity), by situating both in a single formalism based on spectrum evolution equations. The physics of zonal density corrugations (i.e. zonal density structures which distort ∇n) is shown to differ substantially from that of zonal shear flows. This unified analysis addresses the dynamic interplay of different mechanisms and the implications thereof for feedback on turbulence and system states.

This work yielded several new theoretical results worthy of note. These are:

(a) the derivation of a unified set of spectral equations, encompassing non-linear response and polarization and density advection beat noise. Zonal flows and density corrugations are calculated. Table 1 summarize the key theoretical results. Nonlinear invariants are diffused and advected in k_x—space by zonal shear kinetic energy;
(b) vorticity flux correlation is shown to drive zonal flow noise. Likewise, density corrugation noise is driven by density flux correlations. Here 'correlation' refers to two-time correlation. Note it is the flux correlation time which is of interest here;
(c) while the effective viscosity for zonal flows can go negative, the zonal diffusivity remains positive definite for $\alpha > 1$. Thus, DW-ZFT can manifest bi-directional transfer of kinetic energy to large scale with internal energy $\left(\sim\left\langle\left|\frac{\tilde{n}}{n}\right|^2\right\rangle\right)$ to small scale. This is consistent with familiar 2D fluid phenomenon of the dual cascade of potential enstrophy $\left(\sim\left\langle\left|\frac{\tilde{n}}{n}-\rho_s^2\nabla_\perp^2\frac{e\tilde{\phi}}{T}\right|^2\right\rangle\right)$ to small scale, and kinetic energy $\left(\sim\left\langle\left|\rho_s\nabla_\perp\frac{e\tilde{\phi}}{T}\right|^2\right\rangle\right)$ to large scale. The quantity $\langle n^2\rangle$ is the unique inviscid, $\alpha\to 0$ invarient of the Hasegawa–Wakatani system involving only the density field. As there are no other such quadratic invariants involving only density, it tends toward equipartition in k, as in equilibrium statistical mechanics. Physically, this means that the velocity straining field tends to 'chop-up' density, producing smaller scale elements, and accessing

smaller scales until cut-off by resolution or dissipation. Note for $\alpha \to \infty$, density is unaffected and so no inverse cascade occurs, in that limit either. This should be contrasted to the potential field, which is constrained by dual quadratic invariants of energy $\left\langle (\nabla\phi)^2 \right\rangle$ and enstrophy $\left\langle (\nabla^2\phi)^2 \right\rangle$. These force the inverse cascade of energy [66]. This is fundamentally why the viscosity can go negative while the density diffusivity is positive, also a symptom of the absence of inverse cascade;

(d) the effective zonal viscosity can be negative, but need not be! Indeed, the zonal viscosity goes negative only for an energy spectrum which decays sufficiently rapidly in k_r-i.e. $\mu_{eff} < 0$ for $\frac{\partial E}{\partial k_r} < 0$ and $\left|\frac{\partial E}{\partial k_r}\right| < \left|\frac{\partial E}{\partial k_r}\right|_{crit}$. This is consistent with wave kinetics, which links modulational instability to the condition that the slope of the action density in k_r be sufficiently negative. The sensitivity of μ_{eff} to spectral slope re-enforces the importance of treating noise and modulational instability in a unified theory, since the oft-invoked negative viscosity instability (i.e. modulational instability) may be absent or very weak, on account of fluctuation spectrum structure. Table 2 compares wave kinetic and spectral closure results;

(e) the importance of the zonal cross-correlation spectrum $\langle n_k\phi_k^\star\rangle$ was identified. This naturally appears in the statistical theory, and has been heretofore ignored. The spectral closure theory yields the zonal density-potential cross-correlation spectra $\langle n_k\phi_k^\star\rangle$, whose real part turns out to be negative i.e. $\Re\langle n_k\phi_k^\star\rangle < 0$ and the imaginary part $\Im\langle n_k\phi_k^\star\rangle$ vanishes. This means that the zonal density and zonal potential are anti-correlated. This follows from constraining the solution for $\Im\langle n_k\phi_k^\star\rangle$ to be bounded, which constrains that modulational growth rate of zonal intensity be less than the non-linear damping of density corrugation. $\langle n_k\phi_k^\star\rangle$ is significant in all regimes of electron adiabaticity and determines the relative phasing of zonal density and zonal potential. All real space zonal cross-correlations can be determined from zonal density-potential cross spectra $\langle n_k\phi_k^\star\rangle$;

(f) the zonal density and zonal vorticity cross-correlation is $\left\langle \overline{n}\nabla^2\overline{\phi}\right\rangle > 0$, which means that the density peaks are co-located with the vorticity peaks. The density gradient and vorticity gradient cross-correlation becomes positive i.e. $\left\langle \nabla_x\overline{n}\nabla_x^3\overline{\phi}\right\rangle > 0$. This means that the *zonal density jumps are co-located with the zonal vorticity jumps*. Finally, another correlation of interest could be between zonal density gradient and zonal flow $\left\langle -\nabla_x\overline{n}\nabla_x\overline{\phi}\right\rangle$. This can be obtained as $\left\langle -\nabla_x\overline{n}\nabla_x\overline{\phi}\right\rangle = \sum_{k_x} -k_x^2\Re\langle n_k\phi_k^\star\rangle > 0$. This means *density gradient peaks are co-located with the zonal flow peaks*. Zonal correlations appear quite relevant to staircase structure characteristics.

The theoretical results listed above have several immediate pragmatic implications, which this paper explores and develops. These are discussed below.

(a) While polarization beat noise and modulational effects are comparable intrinsically (both set by the Reynolds stress!), the synergy of the two mechanisms is stronger than either alone. This is because zonal noise acts to excite marginally stable and weakly damped zonal flows. It thus expands significantly the range of zonal flow activity relative to that predicted by modulational instability calculations. Noise also increases the branching ratio between zonal flow and turbulence energy.

(b) The interaction of zonal noise and modulations has a significant effect on feed-back processes, and thus the global characteristics of DW-ZFT. Regarding the L→H transition, as described by a simple predator–prey model, we see that noise eliminates the threshold for zonal flow excitation, and so expands the predicted range of the intermediate phase (for all else fixed), while drastically reducing turbulence overshoot. Thus, the nagging question of 'if zonal flows are the L→H trigger, what triggers the trigger?' is eliminated i.e. polarization beat noise triggers the trigger. Since energy transfer to the zonal flow is accelerated, the threshold for L→H transition (which occurs when ∇P steepens, due to a decrease in transport to neoclassical levels) is lowered.

(c) Zonal corrugations are excited by noise, regardless of modulational stability. The zonal density diffusivity is positive definite. Corrugation generation is thus seen a means for seeding transport events.

When reading this rather theoretical paper, the experimentalist (either physical or digital) may ask 'what's in for me?' To this end we note that:

(a) the spectral transfer mechanism for corrugations (i.e. positive diffusivity) is as yet untested. This could be tested using bicoherence analysis of zonal density perturbations and intensity with smaller scale fluctuatios;

(b) the zonal cross-correlation has not been measured, its theory is untested, and its relation to staircase structure has not been addressed. Zonal cross correlation necessitates measurement of the mesoscale ($\sim\sqrt{\rho_i L_i}$) potential structure. In practice, this seems possible only via Heavy Ion Beam Probe [45], or by Langmuir probe, -usually in a limited region near the edge. For either, a measurement might be performed by measuring the density and potential perturbations at low frequency, and on mesoscales, windowing at $k_\theta = 0$ to obtain the zonal component, and then constructing the correlations. Long range correlation analysis [67] is a possibility for this;

(c) the predator-prey dynamics (intensively studied !) drastically changes when zonal noise is accounted for. The domain of zonal flow excitation expands, and the system never reaches the modulational instability threshold;

(d) the improved L–H transition model presented in this paper is eminently testable. In particular, the weak overshoot, expanded domain of zonal mode activity, absence of a 'trigger' modulation and the level of residual H mode turbulence are all seemingly in accord with experimental findings;

(e) one might consider comparisons of 'fits' of I-phase and LH data to the KD'03 model and to it is extension reported here. In particular, the presence or absence of large overshoot, as predicted by KD'03 [4], is one way to discriminate between model versions;
(f) one could examine zonal shear bicoherence with smaller scale fluctuations for the footprint—or lack thereof—of a coherent energy transfer events, symptomatic of a trigger modulation.

More generally, one might:

(a) construct the pdf of zonal flow shears, especially approaching, an in, I-phase;
(b) examine spectral structure for evidence of universality and a critical slope. Some evidence of this short already exists [68].

This paper stimulated several plans for future study. First of these is to understand zonal flow generation when modulational instability is weak or absent. In that case, does shearing occurs in intermittent and bursty avalanche—like feedback events [69]? Does a critical spectral slope self-organize from these interactions? A related question concerns understanding the interaction of corrugations (driven by particle, or more generally heat flux correlations) with avalanches. In particular, we speculate that corrugations occurring in the states of high zonal cross-correlation can be sustained as localized transport barriers, staircase elements, etc by the accompanying shear flow. Likewise, bi-stable systems can sustain a long lived corrugation. However, corrugations occurring in states of low zonal cross-correlation seem likely to overturn, and drive avalanches, as in a running sandpile. This concern seems especially relevant to collisionless trapped electron mode turbulence. Does the density gradient state consist of standing corrugations, running avalanches, or mixtures thereof?

Staircases and layering loom large as topics for further study. Theory should understand the role of noise in staircases, which have been considered only in the context of mean field theory, which neglects fluctuations. One might expect noise to cause an effective increase in turbulence spreading, and so tend to smooth staircase features, along with causing a decrease in the number of steps. Finally, the relation between zonal cross-correlation and staircase structure should be explored. Does the zonal cross-correlation and the physics which governs it set the relative position of corrugation steps and zonal shear layers? Is there an optimal zonal correlation for the staircase state? Note that absolute value cross correlations may also be invoked to infer the staircase pattern. However, spectral closure theory yields spectral cross-correlation $\langle n_k\phi_k^\star\rangle$. So any kind of cross-correlation must be obtained in terms of $\langle n_k\phi_k^\star\rangle$. However to obtain correlation of absolute values in terms of spectral quantities is much more difficult as, it involves convolution of modulii of two infinite series. This can be realized from the following

$$\langle|\nabla_x n|\,|\nabla_x^2\phi|\rangle = \left\langle\left|\sum_{k_{1x}} ik_{1x}n_{k_1}e^{ik_{1x}x}\right|\left|\sum_{k_{2x}} -k_{2x}^2\phi_{k_2}e^{ik_{2x}x}\right|\right\rangle. \tag{116}$$

Now, how to express the right hand side in terms of the spectral cross correlation is not at all obvious. Hence, we defer such a mathematically challenging analysis to future work.

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Appendix A. Derivation of spectral kinetic energy diffusion

It is convenient to define $F_{1p} = \Theta^E_{kpq}\left[a^E_pE_k - a^E_{-k}E_p\right]$ and $F_{2p} = \Theta^E_{kpq}\left[b^E_pR_{nk}E_k - b^E_{-k}R^\star_{np}E_p\right]$. Expanding F_1 around $\vec{p}=\vec{k}$ yields

$$F_{1p} = F_{1k} - q_x\left(\frac{\partial F_{1p}}{\partial p_x}\right)_{\vec{p}=\vec{k}} + \frac{q_x^2}{2}\left(\frac{\partial^2 F_{1p}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}} + \ldots \tag{A1}$$

It is obvious that the real part $F^{(r)}_{1k} = \Re\left(\Theta^E_{kkq}\left[a^E_kE_k - a^E_{-k}E_k\right]\right) = 0$ as $\Re a^E_k = \Re a^E_{-k}$ and $\Im\Theta^E_{kkq} = 0$. Using $q^2 - p^2 = 2q_xk_x - k^2$ one gets

$$\begin{aligned}T^{(1)}_{\phi k} &= \Re\sum_{\vec{p}+\vec{q}=\vec{k}} k_y^2q_x^2k^2E_q \\ &\quad\times\left[-2q_x^2k_x\left(\frac{\partial F_{1p}}{\partial p_x}\right)_{\vec{p}=\vec{k}} - k^2\frac{q_x^2}{2}\left(\frac{\partial^2 F_{1p}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}}\right] \\ &= \Re\sum_{\vec{p}+\vec{q}=\vec{k}}\frac{1}{2}k_y^2q_x^4E_q \\ &\quad\times\left[-4k_xk^2\left(\frac{\partial F_{1p}}{\partial p_x}\right)_{\vec{p}=\vec{k}} - k^4\left(\frac{\partial^2 F_{1p}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}}\right] \\ &= \Re\sum_{\vec{p}+\vec{q}=\vec{k}}\frac{1}{2}k_y^2q_x^4E_q\left[-\frac{\partial}{\partial p_x}p^4\frac{\partial F_{1p}}{\partial p_x}\right]_{\vec{p}=\vec{k}}\end{aligned} \tag{A2}$$

Next

$$\left(\frac{\partial F_{1p}}{\partial p_x}\right)_{\vec{p}=\vec{k}} = \left(\frac{\partial\Theta^E_{kpq}}{\partial p_x}\right)_{\vec{p}=\vec{k}}\left[a^E_kE_k - a^E_{-k}E_k\right]$$

$$+\Theta^E_{kkq}\left[\frac{\partial a^E_p}{\partial p_x}E_k - a^E_{-k}\frac{\partial E_p}{\partial p_x}\right]_{\vec{p}=\vec{k}}$$

so that the real part becomes

$$\left(\frac{\partial F_{1p}}{\partial p_x}\right)^{(r)}_{\vec{p}=\vec{k}} = \Theta^{Er}_{kkq}\left[\frac{\partial a^{Er}_k}{\partial k_x}E_k - a^{Er}_k\frac{\partial E_k}{\partial k_x}\right]. \quad \text{(A3)}$$

Similarly

$$\left(\frac{\partial^2 F_{1p}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}} = \left(\frac{\partial^2\Theta^E_{kpq}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}}\left[a^E_kE_k - a^E_{-k}E_k\right] + 2\left(\frac{\partial\Theta^E_{kpq}}{\partial p_x}\right)_{\vec{p}=\vec{k}}\left[\frac{\partial a^E_p}{\partial p_x}E_k - a^E_{-k}\frac{\partial E_p}{\partial p_x}\right]_{\vec{p}=\vec{k}} + \Theta_{kkq}\left[\frac{\partial^2 a^E_p}{\partial p_x^2}E_k - a^E_{-k}\frac{\partial^2 E_p}{\partial p_x^2}\right]_{\vec{p}=\vec{k}}$$

so that the real part becomes

$$\left(\frac{\partial^2 F_{1p}}{\partial p_x^2}\right)^{(r)}_{\vec{p}=\vec{k}} = \left(\frac{\partial\Theta^{Er}_{kkq}}{\partial k_x}\right)\left[\frac{\partial a^{Er}_p}{\partial k_x}E_k - a^{Er}_k\frac{\partial E_k}{\partial k_x}\right] + \Theta_{kkq}\left[\frac{\partial^2 a^{Er}_k}{\partial k_x^2}E_k - a^{Er}_k\frac{\partial^2 E_k}{\partial k_x^2}\right] = \frac{\partial}{\partial k_x}\left[\Theta^{Er}_{kkq}\left(\frac{\partial a^{Er}_p}{\partial k_x}E_k - a^{Er}_k\frac{\partial E_k}{\partial k_x}\right)\right]. \quad \text{(A4)}$$

Hence the expression for $T^{(1)}_{\phi k}$ becomes

$$T^{(1)}_{\phi k} = \frac{\partial}{\partial k_x}\left[\sum_q \frac{1}{2}k_y^2k^4q_x^4E_q\Theta^{Er}_{kkq}\left(a^{Er}_k\frac{\partial E_k}{\partial k_x} - \frac{\partial a^{Er}_k}{\partial k_x}E_k\right)\right]. \quad \text{(A5)}$$

Similarly F_2 is expanded about $\vec{p}=\vec{k}$,

$$F_{2p} = F_{2k} - q_x\left(\frac{\partial F_{2p}}{\partial p_x}\right)_{\vec{p}=\vec{k}} + \frac{q_x^2}{2}\left(\frac{\partial^2 F_{2p}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}} + \ldots \quad \text{(A6)}$$

where $F_{2k} = \Theta^E_{kkq}\left[b^E_kR_{nk}E_k - b^E_{-k}R^\star_{nk}E_k\right]$ so that it is real part $F^{(r)}_{2k}=0$.

$$\left(\frac{\partial F_{2p}}{\partial p_x}\right)_{\vec{p}=\vec{k}} = \left(\frac{\partial\Theta^E_{kpq}}{\partial p_x}\right)_{\vec{p}=\vec{k}}\left[b^E_kR_{nk}E_k - b^E_{-k}R^\star_{nk}E_k\right] + \Theta^E_{kkq}\left[\frac{\partial b^E_k}{\partial k_x}(R_{nk}E_k) - b^E_{-k}\frac{\partial}{\partial k_x}(R^\star_{nk}E_k)\right]$$

so that the real part

$$\left(\frac{\partial F_{2p}}{\partial p_x}\right)^{(r)}_{\vec{p}=\vec{k}} = \Theta^{Er}_{kkq}\left[\frac{\partial b^E_k}{\partial k_x}(R_{nk}E_k) - b^E_k\frac{\partial}{\partial k_x}(R_{nk}E_k)\right]^{(r)}. \quad \text{(A7)}$$

Similarly

$$\left(\frac{\partial^2 F_{2p}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}} = \left(\frac{\partial^2\Theta^E_{kpq}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}}\left[b^E_kR_{nk}E_k - b^E_{-k}R^\star_{nk}E_k\right] + 2\left(\frac{\partial\Theta^E_{kpq}}{\partial p_x}\right)_{\vec{p}=\vec{k}}\left[\frac{\partial b^E_p}{\partial p_x}R_{nk}E_k - b^E_{-k}\frac{\partial R^\star_{np}E_p}{\partial p_x}\right]_{\vec{p}=\vec{k}} + \Theta_{kkq}\left[\frac{\partial^2 b^E_p}{\partial p_x^2}R_{nk}E_k - b^E_{-k}\frac{\partial^2 R^\star_{np}E_p}{\partial p_x^2}\right]_{\vec{p}=\vec{k}}$$

so that the real part becomes

$$\left(\frac{\partial^2 F_{2p}}{\partial p_x^2}\right)^{(r)}_{\vec{p}=\vec{k}} = \frac{\partial\Theta^{Er}_{kkq}}{\partial k_x}\left[\frac{\partial b^E_k}{\partial k_x}R_{nk}E_k - b^E_k\frac{\partial R_{nk}E_k}{\partial k_x}\right]^{(r)} + \Theta^r_{kkq}\left[\frac{\partial^2 b^E_k}{\partial k_x^2}R_{nk}E_k - b^E_k\frac{\partial^2 R_{nk}E_k}{\partial k_x^2}\right]^{(r)} = \frac{\partial}{\partial k_x}\left[\Theta^{Er}_{kkq}\left(\frac{\partial b^E_k}{\partial k_x}R_{nk}E_k - b^E_k\frac{\partial R_{nk}E_k}{\partial k_x}\right)^{(r)}\right]. \quad \text{(A8)}$$

Eventually it is straightforward to show

$$T^{(2)}_{\phi k} = \frac{1}{k^2}\frac{\partial}{\partial k_x}\left[\sum_q \frac{1}{2}k_y^2k^4q_x^4E_q\Theta^{Er}_{kkq} \times \left(b^E_k\frac{\partial}{\partial k_x}(R_{nk}E_k) - \frac{\partial b^E_k}{\partial k_x}R_{nk}E_k\right)^{(r)}\right]. \quad \text{(A9)}$$

Appendix B. Derivation of spectral internal energy diffusion

It is convenient to define

$$F_{1np} = \Theta_{kpq}\left[d^\star_k\left\langle|n_p|^2\right\rangle - d_p\left\langle|n_k|^2\right\rangle\right]$$

and

$$F_{2np} = \Theta_{kpq}\left[\left(q^2-k^2\right)c_p\left\langle n^\star_k\phi_k\right\rangle - \left(q^2-p^2\right)c^\star_k\left\langle n_p\phi^\star_p\right\rangle\right].$$

Expanding F_1 around $\vec{p}=\vec{k}$ yields

$$F_{1np} = F_{1nk} - q_x\left(\frac{\partial F_{1np}}{\partial p_x}\right)_{\vec{p}=\vec{k}} + \frac{q_x^2}{2}\left(\frac{\partial^2 F_{1np}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}} + \ldots \quad \text{(A10)}$$

It is obvious that the real part $F^{(r)}_{1nk} = \Re\left(\Theta^E_{kkq}\left[d^\star_k\left\langle|n_p|^2\right\rangle - d_p\left\langle|n_k|^2\right\rangle\right]\right) = 0$ as $\Re d^\star_k = \Re d_k$ and $\Im\Theta_{kkq}=0$. Next

$$\left(\frac{\partial F_{1np}}{\partial p_x}\right)_{\vec{p}=\vec{k}} = \left(\frac{\partial\Theta_{kpq}}{\partial p_x}\right)_{\vec{p}=\vec{k}}\left[d^\star_k\left\langle|n_p|^2\right\rangle - d_p\left\langle|n_k|^2\right\rangle\right]_{\vec{p}=\vec{k}} + \Theta_{kkq}\left[d^\star_k\frac{\partial}{\partial p_x}\left\langle|n_p|^2\right\rangle - \frac{\partial d_p}{\partial p_x}\left\langle|n_k|^2\right\rangle\right]_{\vec{p}=\vec{k}}.$$

so that the real part becomes

$$\left(\frac{\partial F_{1np}}{\partial p_x}\right)^{(r)}_{\vec{p}=\vec{k}} = \Theta^{(r)}_{kkq}\left[d^r_k \frac{\partial}{\partial k_x}\left\langle |n_k|^2\right\rangle - \frac{\partial d^r_k}{\partial k_x}\left\langle |n_k|^2\right\rangle\right]. \tag{A11}$$

Similarly

$$\left(\frac{\partial^2 F_{1np}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}} = \left(\frac{\partial^2 \Theta_{kpq}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}}\left[d^\star_k \left\langle |n_p|^2\right\rangle - d_p\left\langle |n_k|^2\right\rangle\right]_{\vec{p}=\vec{k}} + 2\left(\frac{\partial \Theta_{kpq}}{\partial p_x}\right)_{\vec{p}=\vec{k}}\left[d^\star_k \frac{\partial}{\partial p_x}\left\langle |n_p|^2\right\rangle - \frac{\partial d_p}{\partial p_x}\left\langle |n_k|^2\right\rangle\right]_{\vec{p}=\vec{k}} + \Theta_{kkq}\left[d^\star_k \frac{\partial^2}{\partial p_x^2}\left\langle |n_p|^2\right\rangle - \frac{\partial^2 d_p}{\partial p_x^2}\left\langle |n_k|^2\right\rangle\right]_{\vec{p}=\vec{k}}$$

so that the real part becomes

$$\left(\frac{\partial^2 F_{1np}}{\partial p_x^2}\right)^{(r)}_{\vec{p}=\vec{k}} = \frac{\partial \Theta^{(r)}_{kkq}}{\partial k_x}\left[d^r_k \frac{\partial}{\partial k_x}\left\langle |n_k|^2\right\rangle - \frac{\partial d^r_k}{\partial k_x}\left\langle |n_k|^2\right\rangle\right] + \Theta^{(r)}_{kkq}\left[d^\star_k \frac{\partial^2}{\partial k_x^2}\left\langle |n_k|^2\right\rangle - \frac{\partial^2 d_k}{\partial k_x^2}\left\langle |n_k|^2\right\rangle\right] = \frac{\partial}{\partial k_x}\left[\Theta^{(r)}_{kkq}\left(d^r_k \frac{\partial}{\partial k_x}\left\langle |n_k|^2\right\rangle - \frac{\partial d^r_k}{\partial k_x}\left\langle |n_k|^2\right\rangle\right)\right]. \tag{A12}$$

Hence expression for $T^{(1)}_{nk}$ becomes

$$T^{(1)}_{nk} = \frac{\partial}{\partial k_x}\left[\sum_q k_y^2 q_x^4 \left\langle |\phi_q|^2\right\rangle \Theta^{(r)}_{kkq} \left(d^r_k \frac{\partial}{\partial k_x}\left\langle |n_k|^2\right\rangle - \frac{\partial d^r_k}{\partial k_x}\left\langle |n_k|^2\right\rangle\right)\right]. \tag{A13}$$

Notice that, $\left(\frac{\partial F_{1p}}{\partial p_x}\right)^{(r)}_{\vec{p}=\vec{k}}$ did not contribute to $T^{(1)}_{nk}$ as the integrand becomes odd in q_x. Using $q \ll k$, expanding F_2 around $\vec{p}=\vec{k}$ yields

$$F_{2np} = F_{2nk} - q_x\left(\frac{\partial F_{2np}}{\partial p_x}\right)_{\vec{p}=\vec{k}} + \frac{q_x^2}{2}\left(\frac{\partial^2 F_{2np}}{\partial p_x^2}\right)_{\vec{p}=\vec{k}} + \ldots \tag{A14}$$

Now $F_{2nk} = \Theta_{kkq}\left[\left(q^2-k^2\right)c_k\langle n^\star_k\phi_k\rangle - \left(q^2-k^2\right)c^\star_k\langle n_k\phi^\star_k\rangle\right]$ so that its real part $F^{(r)}_{2k}=0$. Similarly, it is straightforward to show that

$$\left(\frac{\partial F_{2np}}{\partial p_x}\right)^{(r)}_{\vec{p}=\vec{k}} = \Theta^{(r)}_{kkq}\left[c_k \frac{\partial}{\partial k_x}k^2\langle n^\star_k\phi_k\rangle - \frac{\partial c_k}{\partial k_x}k^2\langle n^\star_k\phi_k\rangle\right]^{(r)}$$

and

$$\left(\frac{\partial^2 F_{2np}}{\partial p_x^2}\right)^{(r)}_{\vec{p}=\vec{k}} = \Theta^{(r)}_{kkq}\left[c^\star_k \frac{\partial}{\partial k_x}k^2\langle n_k\phi^\star_k\rangle - \frac{\partial c_k}{\partial k_x}k^2\langle n^\star_k\phi_k\rangle\right]^{(r)}.$$

Hence the expression for $T^{(2)}_{nk}$ becomes

$$T^{(2)}_{nk} = \frac{\partial}{\partial k_x}\left[\sum_q k_y^2 q_x^4\left\langle |\phi_q|^2\right\rangle \Theta^{(r)}_{kkq} \times \left(c^\star_k \frac{\partial}{\partial k_x}k^2\langle n_k\phi^\star_k\rangle - \frac{\partial c_k}{\partial k_x}k^2\langle n^\star_k\phi_k\rangle\right)^{(r)}\right]. \tag{A15}$$

Appendix C. Derivation of induced diffusion of spectral total energy and enstrophy at $\hat{\alpha}=\infty$

It is convenient to define

$$f_p = \frac{\sigma^Q_p}{1+p^2}\Theta^Q_{kpq}\left(Q_k - Q_p\right). \tag{A16}$$

Expanding f_p around $\vec{p}=\vec{k}$ yields

$$f_p = f_k - q_x\left(\frac{\partial f_p}{\partial p_x}\right)_{\vec{p}=\vec{k}} + \frac{q_x^2}{2}\left(\frac{\partial^2 f_p}{\partial p_x^2}\right)_{\vec{p}=\vec{k}} + \ldots \tag{A17}$$

Obviously, $f_k=0$. Then using $q^2-p^2=2q_xk_x-k^2$ and the above expansion, one gets

$$T_k = 2\sum_{\vec{q}} k_y^2 q_x^2 \frac{\sigma^Q_k}{1+k^2}k^2 Q_q \left[-2q_x^2 k_x\left(\frac{\partial f_p}{\partial p_x}\right)_{\vec{p}=\vec{k}} - k^2\frac{q_x^2}{2}\left(\frac{\partial^2 f_p}{\partial p_x^2}\right)_{\vec{p}=\vec{k}}\right]. \tag{A18}$$

The derivatives are easily calculated from the expression for f_p above.

$$\left(\frac{\partial f_p}{\partial p_x}\right)_{\vec{p}=\vec{k}} = -\frac{\sigma^Q_k}{1+k^2}\Theta^Q_{kkq}\frac{\partial Q_k}{\partial k_x} \tag{A19}$$

and

$$\left(\frac{\partial^2 f_p}{\partial p_x^2}\right)_{\vec{p}=\vec{k}} = -\frac{\partial}{\partial k_x}\left(\frac{\sigma^Q_k}{1+k^2}\right) - \frac{\partial}{\partial k_x}\left(\frac{\sigma^Q_k}{1+k^2}\Theta^Q_{kkq}\frac{\partial Q_k}{\partial k_x}\right). \tag{A20}$$

Using the derivatives given by equations (A19) and (A20) in equation (A18), after some easy manipulations, yields

$$T_k = \frac{\partial}{\partial k_x}\left[\sum_q k_y^2 q_x^4 Q_q k^4\left(\frac{\sigma^Q_k}{1+k^2}\right)^2\Theta^Q_{kpq}\frac{\partial Q_k}{\partial k_x}\right].$$

Appendix D. Derivation of zonal cross-spectrum equation

The triplet correlations on the right hand side of equation (75) are determined by the phase coherency of the three modes $\vec{k}$, $\vec{p}$, $\vec{q}$. To first order, in a state of turbulence, this phase coherency is determined by the direct interaction among these three modes in the presence of the stochastic background of all other interactions. Denoting the perturbation in ϕ_k due to this direct interaction by $\delta\phi_k$, the triad correlations are approximated as

$$\left\langle n_k\phi_p^\star\phi_q^\star\right\rangle = \left\langle \delta n_k\phi_p^\star\phi_q^\star\right\rangle + \left\langle n_k\delta\phi_p^\star\phi_q^\star\right\rangle + \left\langle n_k\phi_p^\star\delta\phi_q^\star\right\rangle \quad \text{(A21)}$$

$$\langle \phi_k^\star\phi_p n_q\rangle = \langle \delta\phi_k^\star\phi_p n_q\rangle + \langle \phi_k^\star\delta\phi_p n_q\rangle + \langle \phi_k^\star\phi_p\delta n_q\rangle \quad \text{(A22)}$$

$$\langle \phi_k^\star\phi_q n_p\rangle = \langle \delta\phi_k^\star\phi_q n_p\rangle + \langle \phi_k^\star\delta\phi_q n_p\rangle + \langle \phi_k^\star\phi_q\delta n_p\rangle . \quad \text{(A23)}$$

For zonal density beat mode δn_k

$$\left\langle \delta n_k\phi_p^\star\phi_q^\star\right\rangle = \int_{-\infty}^{t} dt' e^{-\eta_k\left(t-t'\right)} \left\langle S_{2k}(t')\phi_p^\star\phi_q^\star\right\rangle$$

where

$$\begin{aligned}\left\langle S_{2k}(t')\phi_p^\star(t)\phi_q^\star(t)\right\rangle &= \hat{z}\cdot\vec{p}\times\vec{q}\left[\left\langle\phi_p(t')n_q(t')\phi_p^\star(t)\phi_q^\star(t)\right\rangle\right.\\ &\quad\left.-\left\langle\phi_q(t')n_p(t')\phi_p^\star(t)\phi_q^\star(t)\right\rangle\right]\\ &= \hat{z}\cdot\vec{p}\times\vec{q}\left[\left\langle\phi_p(t')\phi_p^\star(t)\right\rangle\left\langle n_q(t')\phi_q^\star(t)\right\rangle\right.\\ &\quad\left.-\left\langle\phi_q(t')\phi_q^\star(t)\right\rangle\left\langle n_p(t')\phi_p^\star(t)\right\rangle\right].\end{aligned}$$

Hence

$$\left\langle \delta n_k\phi_p^\star\phi_q^\star\right\rangle = \Theta_{kpq}^\star\hat{z}\cdot\vec{p}\times\vec{q}\left[\left\langle\phi_p\phi_p^\star\right\rangle\left\langle n_q\phi_q^\star\right\rangle - \left\langle\phi_q\phi_q^\star\right\rangle\left\langle n_p\phi_p^\star\right\rangle\right].$$

For zonal potential beat mode $\delta\phi_k$

$$\langle \delta\phi_k^\star\phi_p n_q\rangle = \frac{1}{k_x^2}\int_{-\infty}^{t} dt' e^{-\eta_k\left(t-t'\right)} \langle S_{1k}^\star(t')\phi_p n_q\rangle$$

where

$$\begin{aligned}&\langle S_{1k}^\star(t')\phi_p(t)n_q(t)\rangle\\ &\quad= \hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)\left\langle\phi_p^\star(t')\phi_q^\star(t')\phi_p(t)n_q(t)\right\rangle\\ &\quad= \hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)\left\langle\phi_p^\star(t')\phi_p(t)\right\rangle\left\langle\phi_q^\star(t')n_q(t)\right\rangle.\end{aligned}$$

Hence

$$\langle \delta\phi_k^\star\phi_p n_q\rangle = \frac{1}{k_x^2}\Theta_{kpq}\hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)\left\langle\phi_p^\star\phi_p\right\rangle\left\langle\phi_q^\star n_q\right\rangle$$

and

$$\langle \delta\phi_k^\star\phi_q n_p\rangle = \frac{1}{k_x^2}\Theta_{kpq}\hat{z}\cdot\vec{p}\times\vec{q}\left(q^2-p^2\right)\left\langle\phi_q^\star\phi_q\right\rangle\left\langle\phi_p^\star n_p\right\rangle.$$

So the noise term becomes

$$\begin{aligned}F_{\langle n\phi\rangle_k}^{zonal} &= \sum_{\vec{k}=\vec{p}+\vec{q}}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\frac{1}{k_x^2}\left(q^2-p^2\right)\left[\Theta_{kpq}^\star+\Theta_{kpq}\right]\\ &\quad\times\left[\left\langle\phi_p\phi_p^\star\right\rangle\left\langle n_q\phi_q^\star\right\rangle - \left\langle\phi_q\phi_q^\star\right\rangle\left\langle n_p\phi_p^\star\right\rangle\right].\end{aligned}$$

Now the coherent terms are calculated.

$$\begin{aligned}\left\langle n_k\delta\phi_p^\star\phi_q^\star\right\rangle &= \int_{-\infty}^{t} dt' e^{-\left(-i\omega_p^\star+\eta_p\right)\left(t-t'\right)}\left[a_p^\star\left\langle n_k S_{1p}^\star(t')\phi_q^\star\right\rangle\right.\\ &\quad\left.+b_p^\star\left\langle n_k S_{2p}^\star(t')\phi_q^\star\right\rangle\right]\end{aligned}$$

where

$$\begin{aligned}\left\langle n_k S_{1p}^\star(t')\phi_q^\star\right\rangle &= -\hat{z}\cdot\vec{q}\times\vec{k}\left(k^2-q^2\right)\left\langle n_k(t)\phi_q(t')\phi_k^\star(t')\phi_q^\star(t)\right\rangle\\ &= \hat{z}\cdot\vec{p}\times\vec{q}\left(k^2-q^2\right)\left\langle n_k(t)\phi_k^\star(t')\right\rangle\left\langle\phi_q(t')\phi_q^\star(t)\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle n_k S_{2p}^\star(t')\phi_q^\star\right\rangle &= -\hat{z}\cdot\vec{q}\times\vec{k}\left\langle n_k(t)\phi_q(t')n_k^\star(t')\phi_q^\star(t)\right.\\ &\quad\left.-n_k(t)n_q(t')\phi_k^\star(t')\phi_q^\star(t)\right\rangle\end{aligned}$$

Hence

$$\begin{aligned}\left\langle n_k\delta\phi_p^\star\phi_q^\star\right\rangle &= \Theta_{k,p,q}^\star\hat{z}\cdot\vec{p}\times\vec{q}\left[a_p^\star\left(k^2-q^2\right)\langle n_k\phi_k^\star\rangle\left\langle\phi_q\phi_q^\star\right\rangle\right.\\ &\quad\left.+b_p^\star\left(\left\langle n_k^2\right\rangle\left\langle\phi_q^2\right\rangle - \langle n_k\phi_k^\star\rangle\left\langle n_q\phi_q^\star\right\rangle\right)\right].\end{aligned}$$

Again

$$\begin{aligned}\langle\phi_k^\star\delta\phi_p n_q\rangle &= \int_{-\infty}^{t} dt' e^{-\left(i\omega_p+\eta_p\right)\left(t-t'\right)}\left[a_p\langle\phi_k^\star S_{1p}(t')n_q\rangle\right.\\ &\quad\left.+b_p\langle\phi_k^\star S_{2p}(t')n_q\rangle\right]\end{aligned}$$

where

$$\begin{aligned}\langle\phi_k^\star S_{1p}(t')n_q\rangle &= -\hat{z}\cdot\vec{q}\times\vec{k}\left(k^2-q^2\right)\langle\phi_k^\star(t)\phi_{-q}(t')\phi_k(t')n_q(t)\rangle\\ &= \hat{z}\cdot\vec{p}\times\vec{q}\left(k^2-q^2\right)\langle\phi_k^\star(t)\phi_k(t')\rangle\left\langle\phi_q^\star(t')n_q(t)\right\rangle\end{aligned}$$

and

$$\begin{aligned}\langle\phi_k^\star S_{2p}(t')n_q\rangle &= -\hat{z}\cdot\vec{q}\times\vec{k}\langle\phi_k^\star(t)\phi_{-q}(t')n_k(t')n_q(t)\\ &\quad-\phi_k^\star(t)n_{-q}(t')\phi_k(t')n_q(t)\rangle\\ &= \hat{z}\cdot\vec{p}\times\vec{q}\left[\langle\phi_k^\star(t)n_k(t')\rangle\left\langle\phi_q^\star(t')n_q(t)\right\rangle\right.\\ &\quad\left.-\langle\phi_k^\star(t)\phi_k(t')\rangle\left\langle n_q^\star(t')n_q(t)\right\rangle\right].\end{aligned}$$

Therefore

$$\begin{aligned}\langle\phi_k^\star\delta\phi_p n_q\rangle &= \Theta_{k,p,q}\hat{z}\cdot\vec{p}\times\vec{q}\left[a_p\left(k^2-q^2\right)\langle\phi_k^\star\phi_k\rangle\left\langle\phi_q^\star n_q\right\rangle\right.\\ &\quad\left.+b_p\left(\langle n_k\phi_k^\star\rangle\left\langle n_q\phi_q^\star\right\rangle - \langle\phi_k^\star\phi_k\rangle\left\langle n_q^\star n_q\right\rangle\right)\right]\end{aligned}$$

$$\begin{aligned}\langle\phi_k^\star\delta\phi_q n_p\rangle &= \Theta_{k,p,q}\hat{z}\cdot\vec{p}\times\vec{q}\left[a_q\left(q^2-k^2\right)\langle\phi_k^\star\phi_k\rangle\left\langle\phi_p^\star n_p\right\rangle\right.\\ &\quad\left.+b_q\left(\langle\phi_k^\star\phi_k\rangle\left\langle n_p^\star n_p\right\rangle - \langle\phi_k^\star n_k\rangle\left\langle\phi_p^\star n_p\right\rangle\right)\right].\end{aligned}$$

Next we calculate

$$\begin{aligned}\left\langle n_k\phi_p^\star\delta\phi_q^\star\right\rangle &= \int_{-\infty}^{t} dt' e^{-\left(-i\omega_q^\star+\eta_q\right)\left(t-t'\right)}\left[a_q^\star\left\langle n_k\phi_p^\star S_{1q}^\star(t')\right\rangle\right.\\ &\quad\left.+b_q^\star\left\langle n_k\phi_p^\star S_{2q}^\star(t')\right\rangle\right]\end{aligned}$$

where

$$\begin{aligned}\left\langle n_k\phi_p^\star S_{1q}^\star(t')\right\rangle &= -\hat{z}\cdot\vec{k}\times\vec{p}\left(p^2-k^2\right)\left\langle n_k(t)\phi_p^\star(t)\phi_k^\star(t')\phi_p(t')\right\rangle\\ &= \hat{z}\cdot\vec{p}\times\vec{q}\left(p^2-k^2\right)\langle n_k(t)\phi_k^\star(t')\rangle\left\langle\phi_p^\star(t)\phi_p(t')\right\rangle\end{aligned}$$

and

$$\left\langle n_k\phi_p^\star S_{2q}^\star(t')\right\rangle = -\hat{z}\cdot\vec{k}\times\vec{p}\left\langle n_k(t)\phi_p^\star(t)\left(\phi_k^\star(t')n_p(t')\right.\right.$$

$$
\begin{aligned}
&-\phi_p(t')n_k^\star(t'))\rangle \\
&=\hat{z}\cdot\vec{p}\times\vec{q}\left(\langle n_k(t)\phi_k^\star(t')\rangle\langle n_p(t')\phi_p^\star(t)\rangle\right.\\
&\left.-\langle n_k(t)n_k^\star(t')\rangle\langle\phi_p^\star(t)\phi_p(t')\rangle\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
\langle n_k\phi_p^\star\delta\phi_q^\star\rangle = \Theta_{kpq}^\star\hat{z}\cdot\vec{p}\times\vec{q}&\left[\left(p^2-k^2\right)a_q^\star\langle n_k\phi_k^\star\rangle\langle\phi_p^\star\phi_p\rangle\right.\\
&\left.+b_q^\star\left(\langle n_k\phi_k^\star\rangle\langle n_p\phi_p^\star\rangle-\langle n_kn_k^\star\rangle\langle\phi_p^\star\phi_p\rangle\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_k^\star\phi_p\delta n_q\rangle = \int_{-\infty}^t dt' e^{-(i\omega_q+\eta_q)(t-t')}&\left[c_q\langle\phi_k^\star\phi_pS_{1q}(t')\rangle\right.\\
&\left.+d_q\langle\phi_k^\star\phi_pS_{2q}(t')\rangle\right]
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_k^\star\phi_pS_{1q}(t')\rangle &= -\hat{z}\cdot\vec{k}\times\vec{p}\left(p^2-k^2\right)\langle\phi_k^\star(t)\phi_p(t)\phi_k(t')\phi_p^\star(t')\rangle\\
&=\hat{z}\cdot\vec{p}\times\vec{q}\left(p^2-k^2\right)\langle\phi_k^\star(t)\phi_k(t')\rangle\langle\phi_p(t)\phi_p^\star(t')\rangle.
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_k^\star\phi_pS_{2q}(t')\rangle &= -\hat{z}\cdot\vec{k}\times\vec{p}\left[\langle\phi_k^\star(t)\phi_p(t)\left(\phi_k(t')n_p^\star(t')\right.\right.\\
&\left.\left.-\phi_p^\star(t')n_k(t')\right)\rangle\right]\\
&=\hat{z}\cdot\vec{p}\times\vec{q}\left[\langle\phi_k^\star(t)\phi_k(t')\rangle\langle\phi_p(t)n_p^\star(t')\rangle\right.\\
&\left.-\langle\phi_k^\star(t)n_k(t')\rangle\langle\phi_p(t)\phi_p^\star(t')\rangle\right]
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_k^\star\phi_p\delta n_q\rangle = \Theta_{kpq}\hat{z}\cdot\vec{p}\times\vec{q}&\left[c_q\left(p^2-k^2\right)\langle\phi_k^\star\phi_k\rangle\langle\phi_p\phi_p^\star\rangle\right.\\
&\left.+d_q\left(\langle\phi_k^\star\phi_k\rangle\langle\phi_pn_p^\star\rangle-\langle\phi_k^\star n_k\rangle\langle\phi_p\phi_p^\star\rangle\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_k^\star\phi_q\delta n_p\rangle = \int_{-\infty}^t dt' e^{-(i\omega_p+\eta_p)(t-t')}&\left[c_p\langle\phi_k^\star\phi_qS_{1p}(t')\rangle\right.\\
&\left.+d_p\langle\phi_k^\star\phi_qS_{2p}(t')\rangle\right]
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_k^\star\phi_qS_{1p}(t')\rangle &= -\hat{z}\cdot\vec{q}\times\vec{k}\left(k^2-q^2\right)\langle\phi_k^\star(t)\phi_q(t)\phi_q^\star(t')\phi_k(t')\rangle\\
&=\hat{z}\cdot\vec{p}\times\vec{q}\left(k^2-q^2\right)\langle\phi_k^\star(t)\phi_k(t')\rangle\langle\phi_q(t)\phi_q^\star(t')\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_k^\star\phi_qS_{2p}(t')\rangle &= -\hat{z}\cdot\vec{q}\times\vec{k}\langle\phi_k^\star(t)\phi_q(t)(\phi_q^\star(t')n_k(t')\\
&-\phi_k(t')n_q^\star(t'))\rangle\\
&=\hat{z}\cdot\vec{p}\times\vec{q}[\langle\phi_k^\star(t)n_k(t')\rangle\langle\phi_q(t)\phi_q^\star(t')\rangle\\
&-\langle\phi_k^\star(t)\phi_k(t')\rangle\langle\phi_q(t)n_q^\star(t')\rangle]
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_k^\star\phi_q\delta n_p\rangle = \Theta_{kpq}\hat{z}\cdot\vec{p}\times\vec{q}&\left[c_p\left(k^2-q^2\right)\langle\phi_k^\star\phi_k\rangle\langle\phi_q\phi_q^\star\rangle\right.\\
&\left.+d_p\left(\langle\phi_k^\star n_k\rangle\langle\phi_q\phi_q^\star\rangle-\langle\phi_k^\star\phi_k\rangle\langle\phi_qn_q^\star\rangle\right)\right].
\end{aligned}
$$

Eventually one gets

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}+(\mu+D_n)k_x^2\right)\langle n_k\phi_k^\star\rangle\\
&=2\sum_{\vec{k}=\vec{p}+\vec{q}}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\Theta_{k,p,q}^\star\frac{(q^2-p^2)}{k_x^2}\left[a_p^\star\left(k^2-q^2\right)\langle n_k\phi_k^\star\rangle\right.\\
&\left.\times\langle\phi_q\phi_q^\star\rangle+b_p^\star\left(\langle n_k^2\rangle\langle\phi_q^2\rangle-\langle n_k\phi_k^\star\rangle\langle n_q\phi_q^\star\rangle\right)\right]\\
&+2\sum_{\vec{k}=\vec{p}+\vec{q}}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\Theta_{k,p,q}\left[a_p\left(k^2-q^2\right)\langle\phi_k^\star\phi_k\rangle\langle\phi_q^\star n_q\rangle\right.\\
&\left.+b_p\left(\langle n_k\phi_k^\star\rangle\langle n_q\phi_q^\star\rangle-\langle\phi_k^\star\phi_k\rangle\langle n_q^\star n_q\rangle\right)\right]\\
&+2\sum_{\vec{k}=\vec{p}+\vec{q}}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\Theta_{k,p,q}\left[c_q\left(p^2-k^2\right)\langle\phi_k^\star\phi_k\rangle\langle\phi_p\phi_p^\star\rangle\right.\\
&\left.+d_q\left(\langle\phi_k^\star\phi_k\rangle\langle\phi_pn_p^\star\rangle-\langle\phi_k^\star n_k\rangle\langle\phi_p\phi_p^\star\rangle\right)\right]\\
&+\sum_{\vec{k}=\vec{p}+\vec{q}}\left(\hat{z}\cdot\vec{p}\times\vec{q}\right)^2\frac{1}{k_x^2}\left(q^2-p^2\right)\left[\Theta_{kpq}^\star+\Theta_{kpq}\right]\\
&\times\left[\langle\phi_p\phi_p^\star\rangle\langle n_q\phi_q^\star\rangle-\langle\phi_q\phi_q^\star\rangle\langle n_p\phi_p^\star\rangle\right].
\end{aligned}
\tag{A24}
$$

The first, second and third, square bracketed terms on the right hand side of the above equation are the coherent terms and the last square bracket term is the incoherent noise term.

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